Mathematical Reviews

Vol. 17, No. 11 DECEMBER, 1956 pp. 1169-1262

MISCELLANEOUS

★**Speiser, Andreas. Die geistige Arbeit.** Birkhäuser Verlag, Basel und Stuttgart, 1955. vii+207 pp. 19.25 francs suisses.

Inhalt: Die geistige Arbeit. Die räumliche Deutung der Aussenwelt. Die platonische Lehre vom Unbekannten Gott und die christliche Trinität. Platos Ideenlehre und die Mathematik. Wissenschaft und Glaube. Der Erlösungsbegriff bei Plotin. Die mathematische Betrachtung der Kunst. Die Gottesbeweise von Plato, Anselmus und Leibniz. Die philosophisch-naturwissenschaftliche Fakultät. Geist und Mathematik. Die Grundlagen der Mathematik von Plato bis Fichte. Über die Freiheit. Die mathematische Denkweise in den Geisteswissenschaften und in der Kunst.

Feibleman, James K. Mathematics and its applications in the sciences. Philos. Sci. **23** (1956), 204–215.

Bouligand, Georges. Sur les conditions effectives de la recherche. C. R. Acad. Sci. Paris **242** (1956), 2789–2792.

Gonseth, F. Des mathématiques à la philosophie. Dialectica **9** (1955), 222–243.

Moch, François. Oui, non — peut-être. Dialectica **9** (1955), 244–262.

Au cours d'un travail, quel qu'il soit, la pensée classe les jugements en trois catégories: ceux qu'elle accepte, ceux qu'elle rejette, ceux qu'elle réserve. Elle applique donc une logique à trois valeurs — la „Logique de la Composition" — dont l'étude entraîne des conséquences dans des domaines variés. Dans son domaine propre, elle permet de définir avec précision les instants logiques d'une théorie quelconque. Mais le choix d'une logique détermine le degré de généralité qu'il est possible de donner à la notion d'ensemble; la Logique de la Composition permet d'atteindre la plus grande généralité, qui échappait nécessairement à la Logique bivalente, et elle permet en même temps d'énoncer la relation précise qui unit la Logique et la Théorie des Ensembles. Elle exprime les structures mentales que supposent la Physique probabiliste, et, plus généralement, toute théorie non déterministe — qu'il s'agisse de Physique, de théorie de la connaissance ou de morale. *Résumé de l'auteur.*

Douglas, A. Vibert. Forty minutes with Einstein. J. Roy. Astr. Soc. Canada **50** (1956), 99–102.

Kaplan, Samuel. My tilt with Albert Einstein. American Scientist **44** (1956), 204–211.

van Dantzig, D. Is $10^{10^{10}}$ a finite number? Dialectica **9** (1955), 273–277.

HISTORY, BIOGRAPHY

Haas, Karlheinz. Die mathematischen Arbeiten von Johann Hudde (1628–1704), Bürgermeister von Amsterdam. Centaurus **4** (1956), 235–284.

Tenca, Luigi. Su una svista di stampa in "de Dimensione Parabolae" di Evangelista Torricelli notata da Stefano Angeli. Boll. Un. Mat. Ital. (3) **11** (1956), 258–259.

Fleckenstein, J. O. Der Prioritätsstreit zwischen Leibniz und Newton. Elem. Math. Beiheft no. **12** (1956), 27 pp.

★**Karl Fridrih Gauss, on the centenary of his death (1855-1955). Collection of essays under the general editorship of Academician I. M. Vinogradov.** Izdat. Akad. Nauk SSSR, Moscow, 1956. 311 pp. 11 rubles.

A short scientific biography by Vinogradov is followed by five essays, by B. N. Delone, A. P. Norden, A. I. Markuševič, B. V. Gnedenko and M. F. Subbotin respectively, dealing with the contributions of Gauss to Number Theory, Geometry, Analysis, Probability and Astronomy. There is no index or bibliography.

★**Kagan, V. F. Lobačevskiĭ i ego geometriya: obščedostupnye očerki. [Lobačevskiĭ and his geometry: popular essays].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 303 pp. (1 plate). 6.45 rubles.

This book contains a commemorative oration and five popular essays. The essays are entitled: 1. Theory of parallel lines up to the discovery of non-euclidean geometry. 2. The great savant Lobačevskiĭ and his place in world-science. 3. Elements of non-euclidean geometry in the work of other geometers. 4. Janos Bolyai. 5. Non-euclidean geometry as constructed by Lobačevskiĭ, Gauss and Bolyai.

Hofmann, Jos. E. Über Viètes Konstruktion des regelmässigen Siebenecks. Centaurus **4** (1956), 177–184.

Fried, Ervin; und Varga, Tamás. Über die Frage der Einführung der komplexen Zahlen. Publ. Math. Debrecen **4** (1956), 484–487.

Sudo, Toshiichi. A study of the history of mathematics in Ryu-kyu. III. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **5** (1955), 179–189.

van der Waerden, B. L. Tamil astronomy. Centaurus **4** (1956), 221–234.

Krishna Rav, I. V. M. The motion of the moon in Tamil astronomy. Centaurus **4** (1956), 198–220.

Nádor, György. The Copernican theory and its influences on scientific thought. Magyar Tud. Akad. Mat. Fiz. Tud. Oszt. Közl. **6** (1956), 93–105. (Hungarian)

Sánchez Pérez, José A. Mathematics in the XIV century. Gac. Mat., Madrid (1) **8** (1956), 12–20. (Spanish)

Jitsuo Yoshikawa (1878-1915). J. Gakugei Tokushima Univ. Nat. Sci. Math. **6** (1955), 3–4 (1 plate).

Tsuruichi Hayashi (1873-1935). J. Gakugei Tokushima Univ. Nat. Sci. Math. **6** (1955), 1–2 (1 plate).

Jones, Phillip S. Louis C. Karpinski, historian of mathematics. Science **124** (1956), 19.

Montel, Paul. Nécrologie: Emile Borel. Rev. Gén. Sci. Pures Appl. **63** (1956), 137–141.

Finzi, Bruno. Commemorazione di Alberto Einstein. Ist. Lombardo Sci. Lett. Rend. Parte Gen. Atti Ufficiali (3) **19(88)** (1955), 106–120.

Pleijel, Åke. Ivar Fredholm. Nordisk Mat. Tidskr. **4** (1956), 67–75, 119 (1 plate). (Swedish. English summary)

Faber, Georg. Georg Hamel 12. 9. 1877 - 4. 10. 1954. Bayer. Akad. Wiss. Jbuch. **1955**, 178–180.

★**Kopernik, Nikolaĭ. Sbornik stateĭ i materialov k 410-letiyu so dnya smerti (1543-1953). [Nicolaus Copernicus. Collection of essays and source-materials on the 410th anniversary of his death. 1543-1953.]** Izdat. Akad. Nauk SSSR, Moskow, 1955. 112 pp. 6.40 rubles.

A commemorative address by Academician A. N. Hesmeyanov is followed by four popular essays by A. A. Mihaĭlov, B. F. Poršnev, Academician V. A. Fok and F. Ya. Nesteruk respectively, with the titles: Life and works of Copernicus: The age of Copernicus: The system of Copernicus and the system of Ptolemy in the light of present-day theories of gravitation: Technical works of Copernicus. Finally there is described an exhibition of books and illustrations. The book is beautifully illustrated, especially the last part.

★**Polubarinova-Kočina, P. Ya. Sof'ya Vasil'evna Kovalevskaya: Eë žizn' i deyatel'nost'. [Sof'ya Vasil'evna Kovalevskaya: her life and work.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 100 pp. 1.50 rubles.

A volume in the popular series Lyudi Russkoĭ Nauki (Russian scientists).

Labrador, Juan Francisco. Gregorio Ricci-Curbastro. Gac. Mat., Madrid (1) **8** (1956), 3–5 (1 plate). (Spanish)

Frédéric Riesz, 1880-1956. Acta Math. Acad. Sci. Hungar. **7** (1956), 1–3.

Kertész, Andor. Tibor Szele and his mathematical life-work. Publ. Math. Debrecen **4** (1956), 115–125.

Turnbull, H. W. Obituary: Archibald Read Richardson. J. London Math. Soc. **31** (1956), 376–384.

Dingle, Herbert. Edmund T. Whittaker, mathematician and historian. Science **124** (1956), 208–209.

Julia, Gaston. Notice nécrologique sur Sir Edmund Whittaker. C. R. Acad. Sci. Paris **242** (1956), 2493–2495.

FOUNDATIONS, MATHEMATICAL LOGIC

★**Wittgenstein, Ludwig. Philosophical investigations. Translated by G. E. M. Anscombe.** The Macmillan Co., New York, 1953. x+x+232+232 pp. $6.00.

Translation into English (with the original German text on the left-hand pages) of manuscripts left by Wittgenstein, edited by G. E. M. Anscombe and R. Rhees. An editor's note begins: "What appears as Part I (pp. 1–172) of this volume was complete by 1945. Part II was written between 1947 and 1949. If Wittgenstein had published his work himself, he would have suppressed a good deal of what is in the last thirty pages or so of Part I and worked what is in Part II, with further material, into its place." The author's preface (written in 1945) says in part: "The thoughts ··· concern many subjects: the concepts of meaning, of understanding, of a proposition, of logic, the foundations of mathematics, states of consciousness, and other things. I have written down all these thoughts as "remarks" short paragraphs of which there is sometimes a fairly long chain about the same subject, while I sometimes make a sudden change, jumping from one topic to another."

★**Minsky, M. L. Some universal elements for finite automata.** Automata studies, pp. 117–128. Annals of mathematics studies, no. 34. Princeton University Press, Princeton, N. J., 1956. $4.00.

Taking the finite automata model of Kleene [Automata studies, Princeton, 1956, pp. 3–41; MR **17**, 1040], the author proves that given a function f on a certain class of stimuli to a set of responses, there is a realization of f with some delay d and a realization of f with time expanded by some positive multiplier p in nets which contain only copies of disjunction and conjunction elements plus any element whose response function is not monotonic. The proof makes use of a "unit pulser" by means of which a sequential pattern of stimuli is converted into a non-sequential pattern. *C. Y. Lee* (Chatham, N.J.).

★**Behmann, Heinrich. Muss die Logik paradox sein?** Proceedings of the Second International Congress of the International Union for the Philosophy of Science, Zurich, 1954. vol. II, pp. 97–108. Editions du Griffon, Neuchâtel, 1955.

The author considers logical paradoxes under three heads: (1) those of the propositional calculus; (2) those depending on reflexiveness, (3) those of modal logic. As fundamental to these respective groups he deals with: (1) the paradox of implication; (2) the Russell paradox; (3) the paradox of multiple modalities. He examines previous attempts to eliminate some of the paradoxes, and makes a number of alternative suggestions.

A. Rose (Nottingham).

Bergmann, Gustav. Russell's examination of Leibniz examined. Philos. Sci. **23** (1956), 175–203.

★**Tarski, Alfred. Logic, semantics, metamathematics. Papers from 1923 to 1938.** Translated by J. H. Woodger. Oxford, at the Clarendon Press, 1956. xiv+471 pp. $9.60.

Słupecki, Jerzy. A logical system without operators. Studia Logica 3 (1955), 98–124. (Polish and Russian. English summary)

Adjukiewicz [Studia Philos. 1 (1936), 1–27] suggested how to express quantifiers without the use of operators. This idea is carried out here to build a formal system S in which all variables are free and which embodies the simple theory of types [Tarski, Trav. Soc. Sci. Lett. Varsovie. Cl. III. no. 34 (1933)]. To every theorem of the simple theory of types corresponds a theorem of S with the same meaning (the last expression being recursively defined). S assumes the sentential calculus and two additional axioms: $\varphi=\varphi$, and $\varphi=\psi\rightarrow[\Phi(\varphi)\equiv\Phi(\psi)]$, where φ and ψ are of any (but both of the same) semantical category. Besides the usual rules of detachment and of substitution, S has two special rules. One allows us to accept $\alpha\rightarrow(\Phi=\Psi)$ provided that $\alpha\rightarrow[\Phi(\varphi)\equiv\Psi(\varphi)]$ is a theorem where α is a sentential expression not containing φ. The other is the rule of accepting definitions as equivalences in the system. This rule is in the style of Leśniewski's systems [Fund. Math. 14 (1929), 1–81]. If we introduce the following two definitions: $1(\varphi)\equiv(\varphi=\varphi)$, and $*\langle\varphi_1, \varphi_2, \cdots, \varphi_k\rangle\ (\varphi)\equiv\alpha$ (where α is a concrete sentential formula with $\varphi, \varphi_1, \cdots, \varphi_k$ as its variables and $\varphi_1, \cdots, \varphi_k$ are parameters for $*$), then $*\langle\varphi_1, \varphi_2, \cdots, \varphi_k\rangle=1$ has the same meaning in S as $\Pi\,\varphi\alpha$ has in the simple theory of types. *H. Hiż.*

Cogan, Edward J. A formalization of the theory of sets from the point of view of combinatory logic. Z. Math. Logik Grundlagen Math. 1 (1955), 198–240.

Diese Dissertation gibt zunächst eine Einführung in die kombinatorische Logik, wobei von den Zusammensetzungen aus K und S mit $Kxy=x$, $Sxyz=xz(yz)$ als Kombinatoren ausgegangen wird. Zur Axiomatisierung der Mengenlehre wird ein neuer Kombinator Ξ axiomatisch eingeführt, sodass Ξxy bedeutet $(z)(xz\supset yz)$, d.h. x ist Untermenge vom y. Dann sind Π (Allquantor) und P (Implikation) definierbar. Es wird auch noch Q axiomatisch eingeführt, sodass Qxy bedeutet $x=y$. Da in der kombinatorischen Logik „Zahlen" eingeführt werden können, etwa 0 als KJ, 1 als BJ ($Jx=x$ und $Bxyz=x(yz)$), so ist $Q01$ eine falsche Aussage und die Negation Nx einer Aussage x daher durch $PxQ01$ definierbar.

Anstelle von Typen werden „Kategorien" verwendet und zwar M (Kategorie der Mengen), H (Kategorie der Aussagen), E (Kategorie der Objekte, „obs") und mit α, β stets $F\alpha\beta$ (Kategorie der Abbildungen von α in β). Nach Def. von K und S sind aber Kategorien gegeben durch $F\alpha(F\beta\alpha)$ und $F(F\alpha(F\beta\gamma))(F(F\alpha\beta)(F\alpha\gamma))$, die auffälligerweise die Gestalt der Łukasiewicz-Axiome für die Implikation haben. Es werden anschliessend alle Axiome des Gödelschen Systems kombinatorisch formuliert. Nur für das Unendlichkeitsaxiom und das Auswahlaxiom müssen dabei noch neue Kombinatoren axiomatisch eingeführt werden. *P. Lorenzen* (Kiel).

Carruccio, Ettore. I sistemi quasi coerenti. Boll. Un. Mat. Ital. (3) 11 (1956), 254–256.

Da un' osservazione del Poincaré si prende lo spunto per definire una particolare classe di sistemi razionali, che vengono chiamati quasi coerenti. Si prospettano problemi logici riguardanti detti sistemi, dei quali si propongono diverse applicazioni. *Sunto dell'autore.*

Quine, W. V. On formulas with valid cases. J. Symb. Logic 21 (1956), 148.

The author considers methods of showing a formula of quantification theory to be consistent. He shows that the method in which a valid substitution case is exhibited is available if and only if the formula whose consistency is shown is satisfiable in a one-member universe.

B. A. Galler (Ann Arbor, Mich.).

Grize, Jean-Blaise. L'implication et la négation vues au travers des méthodes de Gentzen et de Fitch. Dialectica 9 (1955), 363–381.

Expository discussion of some results on the interpretation of implication and negation in propositional logic [cf. Curry, A theory of formal deducibility, Univ. of Notre Dame, Indiana, 1950; MR 11, 487; Fitch, Symbolic logic···, Ronald Press, New York, 1952; MR 15, 592].

G. Kreisel (Princeton, N.J.).

Słupecki, Jerzy. S. Leśniewski's calculus of names. Studia Logica 3 (1955), 7–76. (Polish and Russian summaries)

This is the second [see Studia Logica 1 (1953), 44–112; MR 16, 892] paper presenting the work of Leśniewski based mostly on students' notes from his lectures. It deals with ontology. Assuming a logic of quantification, ontology is based on the axiom:

$$\Pi x\Pi y[x\in y\equiv$$

$$(\Sigma z(z\in x)\cdot\Pi z\Pi w((z\in x\cdot w\in x)\supset z\in w)\cdot\Pi z(z\in x\supset z\in y))].$$

The primitive $\in$ differs essentially from $\in$ of the theory of types or from that of set theory; both its arguments are of the same semantical category (type). It is noted that $x\in y$ of ontology can be rendered in type theory as $\iota x\subset y$. Besides definitions similar to those encountered in protothetics, ontology admits definitions that are closures of formulas of the form $x\in *(x_1, \cdots, x_n)\equiv(x\in x\cdot\alpha)$, where $*$ is a new functor, $x_1, \cdots, x_n$ are variables of any semantical category and α a sentential expression with $x_1, \cdots, x_n$ (and possibly x) as its free variables ($n\geqq 0$). Such definitions are creative in the sense that they allow proofs of theorems without defined expressions that are independent of the axiom without a use of the definitions. However, if one adds to the system the definitional axioms that are closures of $\Sigma y(x\in y\equiv(x\in x\cdot\alpha))$, then the definitions cease to be creative. In addition to the extensionality rule similar to that of protothetics, ontology has the rule of extensionality that allows us to accept the closure of any formula of the form

$$\Pi x\Pi x_1\cdots\Pi x_k[x\in\varphi(x_1, \cdots, x_k)\equiv$$

$$x\in\psi(x_1, \cdots, x_k))\equiv\Pi\Phi(\Phi(\varphi)\equiv\Phi(\psi)].$$

Interpreting nominal variables as sentential and $\in$ as conjunction the axiom turns into a theorem of protothetics and the rules are then valid for protothetics. This proves the consistency of ontology. *H. Hiż.*

Grzegorczyk, Andrzej. The systems of Leśniewski in relation to contemporary logical research. Studia Logica 3 (1955), 77–97. (Polish and Russian summaries)

The systems of Leśniewski [see the preceding review] are discussed and compared with works of other authors. Some criticisms are offered, not all of which are fair. The author notes that protothetics involves its own verification by truth-functional procedure and tries to sketch a system of lower functional calculus based on a

somewhat similar idea. However, this is not sufficiently developed. A close affinity is pointed out between the theory of semantical categories and Church's formulation of the simple theory of types [J. Symb. Logic **5** (1940), 56–68; MR **1**, 321]. Ontology is characterized here as the theory of atomic and completely additive Boolean algebras. This is correct so far as the elementary part of ontology is concerned. But the entire ontology is much stronger. Mereology [Leśniewski, Przegląd Filoz. **30** (1927), 164–206; **31** (1928), 261–291; **32** (1929), 60–101; **33** (1930), 77–105; **34** (1931), 142–170] is presented by the author as the theory of Boolean algebras of elements different from the zero element. However, it is doubtful whether the axioms stated by Grzegorczyk provide for the existence of the sum in the mereological sense, i.e., the sum of all those objects x, such that $x \epsilon y$ for a given y. (See, e.g., A4 in Sobociński, Studies in Leśniewski's mereology, Polish Soc. of Arts and Sciences Abroad, Yearbook for 1954–1955, London (1955)). What the author misses is the fact that mereology is based on ontology, just as Leonard and Goodman [J. Symb. Logic **5** (1940), 45–55; MR **1**, 321] base their calculus of individuals on type theory. *H. Hiż.*

Halmos, Paul R. Algebraic logic. I. Monadic Boolean algebras. Compositio Math. **12** (1956), 217–249.

In this paper, which is the first of a series, the author, who is well-known for his elegant expositions of several branches of mathematics, turns his attention to the predicate calculus. By his avowed intention "to make algebra out of logic", he joins a group of logicians which comprises in addition to those mentioned by him (i.e. A. Tarski and some of his collaborators) also L. Henkin, J. Łos, A. Mostowski, and H. Rasiowa. The present paper is concerned with the calculus of one-place predicates (the class calculus).

The algebraic treatment of the propositional calculus within the framework of Boolean algebras is familiar. The additional problem which arises in connection with the class calculus is that of providing an algebraic theory of the quantifiers. These are represented as operators in a Boolean algebra with certain natural properties which are laid down as axioms. The resulting concept is called a monadic algebra. Considerable use is made of M. Stone's representation theory. The dual theory of monadic algebras and of the corresponding Boolean (i.e. totally disconnected compact Hausdorff) spaces is developed in a very interesting manner. The completeness of the class calculus is reflected (in part) in the semi-simplicity of the monadic algebra. Since the elements of a class do not appear explicitly in the theory, individual constants are represented by certain endomorphisms of the algebra. It is proved that every monadic algebra can be extended so that, loosely speaking, a class contains a constant provided the result of applying the existential quantifier to this class holds.

Whenever a branch of mathematics, A, is expounded in terms of another branch, B, the question is raised whether the effort is worthwhile. However, it is an empirical fact that, in addition to its intrinsic interest, such a procedure can benefit both A and B. This does not imply that the older and more autonomous development of A becomes redundant. For the case in hand, it would appear that the "classical approach" is more suitable for the development of the deductive (or axiomatic) theory of the predicate calculus than the procedure of the present paper.

A. Robinson (Toronto, Ont.).

Halmos, Paul R. Predicates, terms, operations, and equality in polyadic Boolean algebras. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 130–136.

This is a preliminary announcement relating to the further development of the work initiated in the paper reviewed above. The algebraic counterpart of the lower predicate calculus is called a polyadic algebra, as distinct from a monadic algebra, which is the author's algebraic version of the class calculus. Polyadic algebras are a good deal more complicated than monadic algebras since they have to provide a theory of n-place predicates (and not only of classes). The author states that his next aim is an algebraic study of Gödel's incompleteness theorem. Details will be published elsewhere. *A. Robinson.*

Rubin, Jean E. Remarks about a closure algebra in which closed elements are open. Proc. Amer. Math. Soc. **7** (1956), 30–34.

The author defines a C-5 algebra as a Boolean algebra with a unary operator C satisfying the Kuratowski axioms on closure and the additional axiom that $-C-Cx=Cx$ for all x; here $-x$ is the complement of x. She observes that this additional axiom is equivalent to several others and that every C-5 algebra can be embedded in a complete one. These observations are made with elegance and with dispatch, but they are not really new; they are essentially contained in Tarski's work on cylindric algebras [cf. Tarski and Jónsson, Amer. J. Math. **73** (1951), 891–939; MR **13**, 426] and the related work of the reviewer on polyadic (and, in particular, monadic) algebras [cf. Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 296–301; MR **15**, 771, and see the paper reviewed second above].

P. R. Halmos (Chicago, Ill.).

Robinson, Abraham. A result on consistency and its application to the theory of definition. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 47–58.

Verf. liefert einen einfachen Beweis für den Bethschen Satz [dieselben Proc. **56** (1953), 330–339; MR **15**, 385] dass eine implizit definierte Relation stets auch explicit definierbar ist, d.h. wird die Aussage

$$(x_1)\cdots(x_k)[F(x_1, \cdots, x_k) \equiv F'(x_1, \cdots, x_k)]$$

von $X(F, G_1, \cdots, G_m) \wedge X(F', G_1, \cdots, G_n)$ impliziert, so gibt es eine formel $Q(x_1, \cdots, x_k)$ die als Relationen nur $G_1, \cdots, G_m$ enthält und für die die Aussage

$$(x_1)\cdots(x_k)[F(x_1, \cdots, x_k) \equiv Q(x_1, \cdots, x_k)]$$

von $X(F, G_1, \cdots, G_m)$ impliziert wird.

Der Beweis ergibt sich aus einem wichtigen Hilfssatz: Ist K eine vollständige Menge von Aussagen (der cl. prädikaten Kalkuls) und sind K_0, K_1 cof. Obermengen von Aussagen, deren gemeinsam vorkommenden Relationen und Individuen schon in K vorkommen, dann ist $K_0 \cup K_1$ cof.

Verf. gibt auch die Spezializierung des Bethschen Satzes auf den Aussagenkalkül an, andererseits die Verallgemeinerung auf den Fall, dass mehrere Relationen $F_1, \cdots, F_n$ zu definieren sind. *P. Lorenzen* (Bonn).

Reichbach, Juliusz. On completeness of the restricted functional calculus. Studia Logica **2** (1955), 213–228. (Polish)

Polish version of Studia Logica **2** (1955), 229–250 [MR **17**, 446]. *H. Hiż* (University Park, Pa.).

Schröter, Karl. Theorie des bestimmten Artikels. Z. Math. Logik Grundlagen Math. **2** (1956), 37–56.

Three possible methods of introducing the definite article (descriptions) into a theory formalized within the lower predicate calculus with identity are as follows. First, the definite article can be introduced by contextual definition schemes. These definition schemes in effect extend the notion of statement of the theory so as to include statements in which descriptive terms appear indicating at the same time how these statements are to be understood as statements in which no descriptive terms appear. Second, the extended notion of statement of the theory can be introduced in the syntactical description of the theory and the definitional schemes of the first method can be introduced as axiom schemes in addition to the axiom schemes of the underlying theory. Lastly, the extended notion of statement can again be introduced immediately but instead of axiom schemes replacing the definition schemes a new rule can be added which permits one to deduce the definiendum from the definiens. The first method is that of Russell, the second method is what the author calls the classical method, and the third method with the rule of deduction modified is that of Hilbert and Bernays.

These three methods are discussed in this paper. There is shown that if the unextended theory is consistent (or complete) then the extended theory resulting from Russell's method is consistent (or complete) and that the elimination theorem is provable for this extended theory. Similar properties are shown to hold for the classical theories by proving that the extended theories resulting from Russell's and the classical methods are deductively equivalent. By making use of these results the elimination theorem for the Hilbert-Bernays theory is given a new simplified proof. *P. C. Gilmore* (University Park, Pa.).

Gumin, Heinz; und Hermes, Hans. Die Soundness der Prädikatenkalküls auf der Basis der Quineschen Regeln. Arch. Math. Logik Grundlagenforsch. **2** (1956), 68–77.

Neuer Beweis für den Satz, dass in dem Kalkül von Quine für die Prädikatenlogik der ersten Stufe [J. Symb. Logic **15** (1950), 93–102; MR **12**, 70] jede Formel die aus den Formeln $A_1, \cdots, A_n$ ableitbar ist, auch eine Folgerung aus $A_1, \cdots, A_n$ im semantischen Sinne ist.

A. Heyting (Amsterdam).

Cobham, Alan. Reduction to a symmetric predicate. J. Symb. Logic **21** (1956), 56–59.

Nach Quine [J. Symb. Logic **19** (1954), 180–182; MR **16**, 208] ist jede quantorenlogisch formulierte Theorie übersetzbar in eine Theorie mit einer einzigen 2-stelligen Relation F. Für dieses F gilt

$$Fxy\cdot \equiv \cdot (\exists z)(x=\{y, z\})\cdot \vee \cdot x=y \wedge Fxx.$$

Hiervon ausgehend beweist Verf., dass anstelle von F sogar stets eine 2-stellige symmetrische Relation gewählt werden kann. *P. Lorenzen* (Kiel).

Montague, Richard; and Henkin, Leon. On the definition of 'formal deduction'. J. Symb. Logic **21** (1956), 129–136.

The relation $\vdash$ is defined between sets of wffs (well-formed formulas) and wffs by the condition that $\Gamma \vdash A$ if and only if a set $D(\Gamma, A)$ is non-empty. The authors list properties which would be desirable in a definition of $D(\Gamma, A)$: (i) there is an effective test to decide, for each decidable set Γ and each wff A, whether a given formal object is or is not a formal deduction of A from assumptions Γ; (ii) $\Gamma \vdash A$ whenever A is an element of Γ or a formal axiom; (iii) if $\Gamma \vdash A$ and $\Gamma \vdash A \supset B$, then $\Gamma \vdash B$; (iv) if $\Gamma \cup \{A\} \vdash B$, then $\Gamma \vdash A \supset B$; (v) if $\Gamma \vdash A$, and a is an individual variable not free in any wff of Γ, then $\Gamma \vdash (a)A$; (vi) if $\Gamma \vdash A$, then $\Gamma \cup \Delta \vdash A$; (vii) if $\Lambda \vdash A$, then A is a formal theorem (where Λ is the empty set); (viii) if $\Gamma \vdash A$, then there is a finite subset Γ_1 of Γ such that $\Gamma_1 \vdash A$. Since, for example, the definition used by A. Church [Introduction to mathematical logic, Part I, Princeton, 1944; MR **6**, 29] fails to have property (vi), another definition of $D(\Gamma, A)$ is proposed, based on the work of S. C. Kleene [Introduction to metamathematics, Van Nostrand, New York, 1952, pp. 94–98; MR **14**, 525], which satisfies (i)–(viii). Properties (ii)–(vii) are shown to be independent and characteristic of the relation $\vdash$. *B. A. Galler.*

Robinson, Abraham. Completeness and persistence in the theory of models. Z. Math. Logik Grundlagen Math. **2** (1956), 15–26.

In this paper the concept of persistence is extended from its meaning in Robinson, J. Symb. Logic **21** (1956), 33–35 [MR **17**, 817] to be applicable to predicates as well as statements of the lower predicate calculus. A counterpart for predicates of the theorem of the mentioned paper is proven here. Further there are introduced two new concepts of completeness. A set K of statements is pre-complete if every universal predicate (in prenex normal form with only universal quantifiers) is persistent with respect to K. There is proven that if K is a pre-complete set then every predicate is persistent with respect to K. A statement or predicate in prenex normal form is said to be of class 0 if it contains no quantifiers and is said to be of class n if it is of class $n-1$ or if in reading the quantifier prefix from one end to the other $n-1$ changes from existential to universal quantifier or universal to existential quantifier occur. A set K of statements is said to be n-complete if for every statement X of class n, constructed only from constants appearing in the statements of K, either X or $\sim X$ is deducible from K. Necessary and sufficient conditions in terms of pre-completeness and 1-completeness are stated and proved for completeness and model-completeness. Finally a necessary and sufficient condition for model-completeness stated in terms of Γ-completeness is proved. *P. C. Gilmore.*

See also: Popadić, p. 1189; Okada, p. 1258.

ALGEBRA

Combinatorial Analysis

Mouette, Léon. Groupements en triades, problème de Kirkman généralisé. Mathesis **65** (1956), 46–52.

Parmi toutes les façons de décomposer $N=6n+3$ éléments en $2n+1$ sous-ensembles de trois, choisir $3n+1$ décompositions telles que toute paire d'éléments figure dans une triade et une seule du tableau complet. L'auteur représente les éléments par les sommets d'un polygone régulier de $N-1$ côtés et par son centre. De la figure représentant le premier bloc de N éléments il déduit les autres blocs par rotation autour du centre. De telle solu-

tions cycliques sont données pour $N=27, 33, 39, 51, 57, 63$. A chaque élément on fait correspondre (suivant une loi qui n'est pas précisée) un nombre écrit dans le système binaire. Trois éléments appartiennent à une même triade du bloc initial si la somme de leurs nombres, chiffre à chiffre, sans faire les retenues, est composé de chiffres pairs. Des remarques sont faites sur les solutions, sans énoncé général. *A. Sade* (Marseille).

Yamamoto, Koichi. Structure polynomial of Latin rectangles and its application to a combinatorial problem. Mem. Fac. Sci. Kyusyu Univ. Ser. A. **10** (1956), 1–13.

J. Touchard [C. R. Acad. Sci. Paris **198** (1934), 631–633] a indiqué le nombre des permutations de n chiffres $1, 2, 3, \cdots, n$ qui sont discordantes avec deux permutations données P et Q. Un cas particulier important est celui où $P=1, 2, \cdots, n$; $Q=2, 3, \cdots, n, 1$. Plus généralement, soit à dénombrer les permutations qui sont discordantes avec les k permutations $P_1, P_2, \cdots, P_k=k, k+1, k+2, \cdots, k-1$, qui forment un rectangle latin. Soit $L_1, L_2, \cdots$ une suite de rectangles latins, posons $L=L_k, L_{k+1}, \cdots, L_n, \cdots$; $Q_n(\mathbf{E})=f(L_n; \mathbf{E}-k)$; $G_k(x, \mathbf{E})=\sum Q_n(\mathbf{E})x^n$, où $\mathbf{E}$ désigne l'opérateur de déplacement usuel: $\mathbf{E}g(x)=g(x+h)$. L'opérateur $H(f(x))=m-xf'(x)/f(x)$, m=degré de $f(x)$, $f(x)\neq 0$; $f'(x)$=dérivée de $f(x)$, projette homomorphiquement le groupe multiplicatif des fonctions rationnelles $K(x)^*\neq 0$ sur le groupe additif des mêmes fonctions. Une H-fonction est une fonction image dans cet homomorphisme, et une h-fonction est la somme d'un polynome et de H-fonctions. Cela posé, l'auteur conjecture que, $G_k(x, \mathbf{E})$ est une h-fonction pour toute valeur de k. Il en déduit diverses propriétés, une formule récursive, une expression asymptotique, et retrouve les résultats de Touchard pour $k=2$, ainsi que des expressions plus simples que les formules connues pour $k=3$. Tables pour $k=3$ et $n=3, 4, \cdots, 20$. *A. Sade* (Marseille).

Johnson, Paul B. Stacking colored cubes. Amer. Math. Monthly **63** (1956), 392–395.

The six faces of a cube can be colored with six given colors in thirty ways. In a set of thirty equal cubes all colored differently, any one, say K, determines eight others which can be stacked to form a large cube colored like K, with the rule that, where any two of the component cubes touch each other, the faces in contact are colored alike [P. A. MacMahon, Nature **47** (1893), 406; F. Winter, Das Spiel der 30 bunten Würfel, Teubner, Leipzig-Berlin, 1934]. The author waives the rule about contact, and then finds that, for each K, the "eight others" can be selected from the remaining 29 in 67260 different ways, and that, for each of these selections, the stacking can be done in 2, 4, 8 or 16 ways. *H. S. M. Coxeter* (Toronto, Ont.).

Zierler, Neal. On a class of binary sequences. Proc. Amer. Math. Soc. **7** (1956), 675–681.

Sequences are treated in which the elements $x(i, j)$ may be either 0 or 1 and in which $i=1, \cdots, n$ and $j=1, 2, \cdots$. For fixed i the sequence is assumed to have a least positive period p_i in j where the integers $p_1, \cdots, p_n$ are pairwise relatively prime. A dependent sequence $f\langle x(1, j), \cdots, x(n, j)\rangle$ is constructed where f is a Boolean function of n variables. This theory has application to the problem of forming a sequence of pseudo-random digits in a computer by combining a set of periodic signals. The author counts the number of distinct sequences f which may be formed if one is allowed to vary the Boolean function and the independent sequences $x(i, j)$ but must retain fixed periods $p_1, \cdots, p_n$. Those circumstances are determined under which identical dependent sequences are obtained from different independent sequences. *D. E. Muller* (Urbana, Ill.).

Rényi, Alfréd. On a combinatorial problem. Mat. Lapok **6** (1955), 151–164. (Hungarian. Russian and English summaries)

The following problem arose in connection with the crossing of types of alfalfa: let $M(n, r)$ denote the number of elements of the shortest sequence formed from the integers $1, \cdots, n$ that is such that each pair of integers (i, j), $1\leq i<j\leq n$, occurs at least once in positions in which they are separated by not more than $r-1$ elements of the sequence. The paper contains some inequalities concerning $M(n, r)$ and some constructions. It is clear that $M(n, 1)=\binom{n}{2}+1$, if n is odd, and $M(n, 1)=\binom{n}{2}+\frac{1}{2}n$, if n is even.

It is shown that

$$\frac{1}{2r}\leq \liminf_{n\to\infty}\frac{M(n, r)}{n^2}\leq \limsup_{n\to\infty}\frac{M(n, r)}{n^2}\leq\frac{1}{r+1}.$$

It is known that $\lim M(n, 2)/n^2=\frac{1}{4}$ (N. G. de Bruijn, unpublished); however, the existence of the limit for $r=3, 4, \cdots$ is an open question. *H. W. Kuhn.*

See also: Meinardus, p. 1187; Freund and Pozner, p. 1221; Kruskal, p. 1231.

Linear Algebra, Polynomials, Invariants

Cherubino, Salvatore. Logaritmi e radici r^{mo} di matrici non singolari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 24–29.

The logarithm of a matrix A being a solution of the matricial equation $e^X=A$, the author has defined its principal value ($\lg A$) for a non-singular matrix A in a previous paper [Rend. Mat. e Appl. (5) **14** (1954), 221–238; MR **16**, 665; see also Rasch, J. Reine Angew. Math. **171** (1934), 65–119, pp. 71–74]. To take care of the multiple valuedness of $\log A$, it is shown that if I is the identity matrix and $\log I=2\pi iD$, then D is any matrix equivalent to a diagonal matrix, having integers along the diagonal. As a consequence if $\log A=\lg A+2\pi iY$, then $Y=D_A$ is of the character of D and commutative with A. This makes possible a definition of A^B, where A is a non-singular matrix and B is of the same order as A and permutable with it, as well as the solution of the matricial equation $X^r=A$, where r is positive integer >1.
T. H. Hildebrandt (Ann Arbor, Mich.).

Hodges, John H. Exponential sums for skew matrices in a finite field. Arch. Math. **7** (1956), 116–121.

Put

$$W(B, r)=\sum_C e\{-\sigma(BC)\},$$

where B, C denote skew matrices of order t with elements in $\mathrm{GF}(q)$, $q=p^n$, $p>2$, and the summation is restricted to matrices of rank $2r$; $\sigma(BC)$ denotes the trace of BC and for $\alpha\in\mathrm{GF}(q)$,

$$e(\alpha)=\exp\{2\pi it(\alpha)/p\},\quad t(\alpha)=\alpha+\alpha^p+\cdots+\alpha^{p^{n-1}}.$$

The main result of the paper consists of an explicit

evaluation of the sum $W(B, r)$, namely

$$(*)\ W(B,r)=q^{jr}\sum_{k=0}^{r}(-1)^k q^{k(k-2j-1)}\begin{bmatrix} j \\ k \end{bmatrix}' S(t-2j,\ 2r-4k),$$

where B is of rank $2j$,

$$S(t,\ 2k)=q^{k(k-1)}\frac{\prod_0^{2k-1}(q^{t-i}-1)}{\prod_1^{k}(q^{2i}-1)},$$

$$\begin{bmatrix} m \\ k \end{bmatrix}'=\frac{(1-q^{2m})\cdots 1-q^{2((m-k+1))}}{(1-q^2)\cdots(1-q^{2k})};$$

note that $S(t, 2k)$ is the number of skew matrices of order t and rank $2k$ in GF(q). [For the result corresponding to (*) in the case of Hermitian matrices and also of general rectangular matrices, see Carlitz and Hodges, Duke Math. J. **22** (1955), 393–405; Hodges, ibid. **22** (1955), 497–509; the symmetric case is treated in a forthcoming paper.) The sum in the right member of (*) is also expressed in terms of a basic hypergeometric series of type ${}_5\Phi_5$.

Application of the function $W(B, r)$ is made to (1) the determination of the number of $m\times t$ matrices X such that $X'AX=B+Y$, where A and B are given skew matrices, A is non-singular of order $2m$, B is of order t, and Y is skew of order t and rank $(2r)$; (2) evaluation of the number of ways in which a given skew matrix A of order t can be partitioned into the sum of k skew matrices of order t and rank $2r_1, \cdots, 2r_k$. *L. Carlitz.*

Perron, Oskar. Über die Abhängigkeit von Potenzsummen und einen Satz von Pólya. Math. Z. **63** (1955), 19–30.

Perron, Oskar. Über Potenzsummen. Math. Z. **64** (1955), 103–114 (1956).

Let K be a field of characteristic 0, let $x_1, \cdots, x_n$ be indeterminates and let S be the field of all symmetric rational functions of $x_1, \cdots, x_n$ with coefficients in K. If $s_\nu=x_1^\nu+\cdots+x_n^\nu$ is the νth power sum of the x_i, $\nu=1, 2, \cdots$, it was shown by Borchardt [Monatsber. Preuss. Akad. Wiss. **1857**, 301–311] that

$$S=K(s_1, s_3, \cdots, s_{2n-1}).$$

Later, Vahlen [Acta Math. **23** (1900), 91–120] generalized this as follows. Let k be an integer $\geqq 2$ and let $\nu_1, \nu_2, \cdots, \nu_n$ denote the first n natural integers which are not multiples of k. Then $S=K(s_{\nu_1}, \cdots, s_{\nu_n})$. Finally, Kakeya [Jap. J. Math. **2** (1925) 69–80; **4** (1927) 77–85] gave necessary and sufficient conditions that the equation $S=K(s_{\nu_1}, \cdots, s_{\nu_n})$ holds for n natural integers $\nu_1, \cdots, \nu_n$. The author proves Borchardt's and Vahlen's results again using a more constructive procedure. In particular, he studies the formulas expressing the elementary symmetric functions of $x_1, \cdots, x_n$ by $s_{\nu_1}, \cdots, s_{\nu_n}$. The functions $s_1, s_3, \cdots, s_{2n+1}$ are algebraically dependent; the corresponding algebraic relation is investigated. If K is the field of complex numbers and if $c_1, c_2, \cdots, c_n$ are complex numbers, the solutions of the equations $s_{\nu_1}=c_1, \cdots, s_{\nu_n}=c_n$ are considered and their multiplicities are determined. *R. Brauer* (Cambridge, Mass.).

Marinescu, G. Espaces vectoriels ordonnés ayant les coefficients dans une algèbre. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1952), no. 1, 17–19. (Romanian. Russian and French summaries)

Two trivial theorems about partially ordered algebras and linear spaces over such algebras. *E. Hewitt.*

Garrett, James Richard. Reduction of equations to normal form in fields of characteristic p. Duke Math. J. **23** (1956), 241–251.

An equation $x^n+a_1x^{n-1}+\cdots+a_n=0$ over a field of characteristic p is reduced to a principal equation $y^n+b_3y^{n-3}+\cdots+b_n=0$ or else to a normal equation $z^n+c_4z^{n-4}+\cdots+c_n=0$ by a Tschirnhaus transformation putting y, or z, equal to $u_1x^{n-1}+u_2x^{n-2}+\cdots+u_n$. The parameters $u_1, u_2, \cdots, u_n$ are then determined in such a way that the coefficients of y^{n-1} and y^{n-2}, or those of z^{n-1}, z^{n-2} and z^{n-3}, are zero.

The case of a quintic having been dealt with in a previous paper [same J. **18** (1951), 373–384; MR **13**, 6] the present investigation is concerned with equations of degrees exceeding 5.

Different cases arise according as $p=2, 3, 5$ or $p\geqq 7$, and again according as $p\nmid n$ or $p|n$. *W. Ledermann.*

Occhini, Luiz. Beitrag zu Walls Verfahren der getrennten Berechnung von Real- und Imaginärteilen der Nullstellen eines Polynoms. Z. Angew. Math. Mech. **36** (1956), 139–145. (English, French and Russian summaries)

By the use of Orlando's "half-sum polynomials" [Math. Ann. **71** (1912), 233–245] the reviewer's method [Bull. Amer. Math. Soc. **52** (1946), 144–157, 890–898; MR **7**, 295; **8**, 154] for computing the zeros of a polynomial is modified. Furthermore, it is shown that the constant term of the half-sum polynomial is proportional to the Hurwitz determinant of the original polynomial, and by the method of Wall [Analytic theory of continued fractions, Van Nostrand, New York, 1948; MR **10**, 32] and Frank [loc. cit.] the Hurwitz determinant can be computed by continued fractions. *E. Frank.*

Heinrich, H. Zur Vorbehandlung algebraischer Gleichungen. Z. Angew. Math. Mech. **36** (1956), 145–148.

A scheme is set up for the computation of the Sturm sequence of a real polynomial of degree n and one of degree $n-1$. *E. Frank* (Chicago, Ill.).

Ostrowski, Alexander. Mathematische Miszellen. XXIV. Zur relativen Stetigkeit von Wurzeln algebraischer Gleichungen. Jber. Deutsch. Math. Verein. **58** (1956), Abt. 1, 98–102.

Let

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_n,\ a_n\neq 0,$$

$$g(z)=z^n+b_1z^{n-1}+\cdots+b_n,\ b_n\neq 0.$$

Denote by r_j and s_j respectively the moduli of the zeros of $f(z)$ and $g(z)$, with $r_1\leqq r_2\leqq\cdots\leqq r_n$, $s_1\leqq s_2\leqq\cdots\leqq s_n$, and let $k>0$ with $4nk^{1/n}<1$. Let the Newtonian majorant $M(f)$ of $f(z)$ be defined as the polynomial

$$M(f)=T_0z^n+T_1z^{n-1}+\cdots+T_n$$

with the smallest positive coefficients T_j such that $|a_j|\leqq T_j$, $T_j^2\geqq T_{j-1}T_{j+1}$. The author shows that, if both $M(kf)$ and $M(kg)$ are both simultaneously majorants of the polynomial $h(z)=g(z)-f(z)$, then for $j=1, 2, \cdots, n$, $B^{-m}\leqq(s_j/r_j)\leqq B^m$, where $B=(1+k^{1/n})/(1-k^{1/n})$ and m is the largest odd integer contained in n. Thus, if $k\downarrow 0$,

$$|(s_j/r_j)-1|=O(2nk^{1/n}).$$

The proof depends upon methods and results given in A. Ostrowski, Acta Math. **72** (1940), 99–257, [MR **1**, 323; **2**, 342]. *M. Marden* (Milwaukee, Wis.).

Vasil'eva, N. K. On searching for the common roots of two algebraic equations. Irkutsk. Gos. Univ. Trudy. **8** (1953), no. 1, 18–21. (Russian)

An elementary exercise in symmetric functions.

E. R. Kolchin (New York, N.Y.).

Sokolov, N. P. On invariants of a cubic ternary form over the field of real numbers. Ukrain. Mat. Ž. **6** (1954), 282–294. (Russian)

The author considers the coefficients A_{ikl} of the cubic form

$$f=\sum_{i,k,l} A_{ikl}x_ix_kx_l$$

as the elements of a cubic matrix. From this matrix he obtains by sections the cubic matrix of the Hessian H. Considering minors he arrives at two other matrices $C_{jk}^{(m)}$ and $K_{jk}^{(m)}$. The elements $K_{jk}^{(m)}$ are given in full at the end of the paper. The author considers his representation as much simpler than that of Aronhold [J. Reine Angew. Math. **55** (1858), 97–191]. Whatever may be the case, written in Clebsch-Weitzenböck symbols the author's representations correspond to

$$f=(a'x)^3,\ H=(a'b'c')^2(a'x)(b'x)(c'x),$$

$$C\sim(a'b'u')^2(a'x)(b'x),\ K\sim(a'H'u')^2(a'x)(H'x),$$

and after Weitzenböck succeeded [Akad. Wetensch. Amsterdam Proc. **30** (1927), 166–170] in proving the second fundamental theorem of the symbolic method for complex symbols also, up till now explicit representations, e.g. as that for $K_{jk}^{(m)}$ given by the author, were generally considered to be without interest. *E. M. Bruins.*

Sokolov, M. P. On the application of spatial matrices to the investigation of cubic ternary forms over the field of real numbers. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 159–164 (1954). (Ukrainian. Russian summary)

This paper contains the results and methods used in classifications of ternary cubic forms. These are given more in detail in the papers hereafter. *E. M. Bruins.*

Sokolov, N. P. Projective classification of cubic ternary forms in the real domain. Ukrain. Mat. Ž. **6** (1954), 405–417. (Russian)

The author classifies first the non-degenerated cubic forms starting from the canonic form $x^3+y^3+z^3+6mxyz$; he writes in full the seven numerically rather complicated transformations of the cubic matrix in the case $T=0$, $0<m<1$; $T=0$, $m<-1$; $S=0$, $m=1$, where T and S denote the Clebsch invariants of the cubic form. If one takes into account that the simple ternary transformation

$$x=x'+y'+z';\ y=-x'+y'+z';\ z=-\frac{6m}{1+2m^3}y'+\frac{1}{m}z',$$

transforms this cubic form into (apart from a factor)

$$2m^3(8m^6+20m^3-1)y^3+(1+2m^3)^3z^3+$$
$$6(1+2m^3)^2m^3x^2y+12m^3(1-m^3)yz^2$$

which gives for $T=c(8m^6+20m^3-1)=0$ and for $m=1$ directly three standard forms which the author obtains, one is led to the question what the particular interest and advantage of the use of three dimensional matrices is.

According to the same methods the author obtains standard forms for cubics with nodes, cusps and the various degenerations of cubics in conics and straight lines. Clebsch's analysis [Vorlesungen über Geometrie, Bd. I, Teubner, Leipzig, 1876] seems to the reviewer to be much simpler and from the geometrical description all standard forms can be written down straightforward in the degenerate cases.

The author gives in full the matrices C, K, for his 13 standard forms. He does not remark that, in case the Hessian is proportional to the form, C and K are also proportional! In any case he gives explicitly all 81 elements of these matrices. *E. M. Bruins* (Amsterdam).

Sokolov, N. P. Projective classification of real plane curves of third order. Ukrain. Mat. Ž. **7** (1955), 295–304.

This classification follows immediately from the paper reviewed above, in which canonical forms of all real cubic ternary forms F with respect to the group of real projective transformations are given.

When discriminant $F=0$, the canonical form is $x_1^3+x_2^3+x_3^3+6mx_1x_2x_3$, $m=-\frac{1}{2}$, and when discriminant $F=0$, there are 13 canonical forms all without parameters. In the present paper the geometrical meaning of m is studied, and the other 13 types of cubic curves are described. The results are contained in two tables.

F. J. Terpstra (Pretoria).

See also: Amitsur, p. 1179; Drazin, p. 1179; Kastler, p. 1260.

Lattices

Kogan, S. A. Solution of three problems in lattice theory. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 185–190. (Russian)

Theorem. An element a of a complete lattice L is isolated in the order topology if and only if it is not the least upper bound or greatest lower bound of any chain in $L-a$. This solves one half of Problem 21 of G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; MR **10**, 673]. The author also solves the other half — isolation in the interval topology — but appears unaware of a similar solution by E. S. Northam [Proc. Amer. Math. Soc. **4** (1953), 824–827; MR **15**, 244]. Theorem. For a lattice to be a Hausdorff space in the interval topology, it is necessary and sufficient that for each pair a_1, a_2 of distinct elements of L, the complement A of the sum of the closed intervals $[-\infty, a_1\cap a_2]$ and $[a_1\cup a_2, +\infty]$ has a finite subset such that each element of A is comparable with at least one element of the subset. This solves Problem 23 of Birkhoff. Stated are necessary and sufficient conditions for each closed ideal of a lattice to be an intersection of principal ideals; this solves Birkhoff's Problem 24. As the author notes, this problem has already been declared trivial [Birkhoff, Proc. Internat. Congress Math. Cambridge, Mass., 1950, vol. 2, Amer. Math. Soc., Providence, R.I., 1952, pp. 4–7; MR **13**, 718]; in fact, it is even more trivial than appears here, since the principal condition stated is redundant [cf. J. Hashimoto, Math. Japon. **2** (1952), 149–186, but Hashimoto seems to have overlooked the necessity of the lattice having a greatest element; MR **15**, 192].

P. M. Whitman (Silver Spring, Md.).

Fulkerson, D. R. Note on Dilworth's decomposition theorem for partially ordered sets. Proc. Amer. Math. Soc. **7** (1956), 701–702.

A theorem due to the reviewer [Ann. of Math. (2)

51 (1950), 161–166; MR **11**, 309] asserts that the minimal number of chains into which a partially ordered set can be decomposed is equal to the maximal number of non-comparable elements. In this note, the author shows that a special case of this theorem (namely, for finite partially ordered sets) is equivalent to a theorem of König on graphs which asserts that the maximal number of join arcs is equal to the minimal number of cut nodes. The proof of equivalence is shorter and simpler than the known proofs of either of the two theorems. *R. P. Dilworth.*

Havel, Václav. The decompositions of elements of the lattice with minimal condition. Časopis Pěst. Mat. **80** (1955), 1–16. (Czech. Russian and English summaries)

Let S be a lattice satisfying the descending chain condition. Let ϱ be a symmetric binary relation on S. The join $a_1 \vee a_2 \vee \cdots \vee a_n = c$ $(n > 1)$ is called a ϱ-decomposition of the element c if $a_i \varrho (a_{k_1} \vee a_{k_2} \vee \cdots \vee a_{k_j})$ holds for every $i = 1, 2, \cdots, n$ and for every choice of the various $k_1, k_2, \cdots, k_j$ from the set $\{1, \cdots, i-1, i+1, \cdots, n\}$. This definition defines the proper decomposition for $\varrho = \varrho_1$ where $x \varrho_1 y$ if and only if x and y are incomparable, the direct decomposition for $\varrho = \varrho_2$ where $x \varrho_2 y$ if and only if $x \varrho_1 y$ and $x \wedge y = 0$, the strong decomposition for $\varrho = \varrho_3$ where $x \varrho_3 y$ if and only if $x \varrho_1 y$ and $(x \vee p) \wedge (y \vee p) = p$ for every p in S.

A sufficient condition is given for every ϱ-decomposable element in S to have a ϱ-decomposition with ϱ-indecomposable factors, and a sufficient condition for the uniqueness of this decomposition. In a modular lattice, the first condition is satisfied by the relation ϱ_2, the second by the relation ϱ_3.

The reviewer's note: The formulation of theorem 5 of the appendix is confused. *M. Novotný* (Brno).

Kolibiar, Milan. On the relation "between" in lattices. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5** (1955), 162–171. (Slovak. Russian summary)

The author studies two ternary relations G, M in a lattice S which are defined as follows: 1. $a \times b\, G$ if and only if $(a \cap x) \cup (b \cap x) = x = (a \cup x) \cap (b \cup x)$; 2. $a \times b\, M$ if and inly if $a \cap b \leq x \leq a \cup b$. Let $G(a, b)$, $M(a, b)$ respectively, be the set of all $x \in S$ for which $a \times b\, G$, $a \times b\, M$ respectively, holds. Then $G(a, b) = M(a, b)$ holds if and only if the interval $\langle a \cap b, a \cup b \rangle$ is isomorphic with the cardinal product $\langle a \cap b, a \rangle \times \langle a \cap b, b \rangle$ in such a manner that the elements $(a, a \cap b)$, $(a \cap b, b)$ respectively, correspond to the elements a, b respectively. If the intervals $\langle a \cap b, b \rangle$, $\langle a, a \cup b \rangle$ are contained in $G(a, b)$, then the correspondences $x \to a \cup x$ $(x \in \langle a \cap b, b \rangle)$, $y \to b \cap y$ $(y \in \langle a, a \cup b \rangle)$ are the inverse isomorphisms between $\langle a \cap b, b \rangle$ and $\langle a, a \cup b \rangle$. This is a generalization of the Theorem 6, Chap. V of Birkhoff's book [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; MR **10**, 673]. In a modular lattice $G(a, b)$ is isomorphic with the cardinal product $\langle a \cap b, a \rangle \times \langle a \cap b, b \rangle$.

Let S_1, S_2 be two lattices which are defined on the same set M. Then the following propositions are equivalent: (B) If the set X forms a convex sublattice in S_1, then X forms a convex sublattice in S_2, and vice versa. (D) There exist two lattices A, B defined on the sets M_1, M_2 respectively, and a one-to-one correspondence between M and $M_1 \times M_2$ which is an isomorphism between S_1 and $A \times B$ and simultaneously an isomorphism between S_2 and $\bar{A} \times B$ ($\bar{A}$ is the dual of A). (G) If $a \times b\, G$ holds in S_1, then $a \times b\, G$ holds in S_2, and vice versa. (H) If $a \times b\, M$ holds in S_1, then $a \times b\, M$ holds in S_2, and vice versa.

The reviewer's note: The relations G, M have been studied by Pitcher and Smiley [Trans. Amer. Math. Soc. **52** (1942), 95–114; MR **4**, 87] while the author affirms incorrectly that the relation G is due to Gel'fand. Theorem 1.1 is known [Duthie, ibid. **51** (1942), 1–14, Th. 2.8; MR **4**, 74]. *M. Novotný* (Brno).

Jakubík, Ján. Congruence relations and weak projectivity in lattices. Časopis Pěst. Mat. **80** (1955), 206–216. (Slovak. Russian and English summaries)

Let $i_0, i_1, \cdots, i_n$ be intervals in a lattice S; let the interval i_k be contained in a interval i_k' such that i_k' and i_{k-1} are transposes $(k = 1, 2, \cdots, n)$; thus the interval i_0 is called weakly projective to the interval i_n (in symbols $i_0 \succeq i_n$). A lattice S is called discrete, if all chains with the least and the greatest element in S are finite. The set Y of all prime intervals of a discrete lattice S is quasi-ordered by the relation $\succeq$; by identifying all the prime intervals p, q in Y, for which $p \succeq q$, $q \succeq p$, we obtain a partly ordered set X. Then 2^X is anti-isomorphic with the set of all congruence relations of the discrete lattice S, the relations being ordered by the usual manner [Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 3; MR **10**, 673]. This is a solution of the Problem 67 of Birkhoff [loc.cit. p. 144]. The congruence relations on a discrete lattice S form a Boolean algebra if and only if the relation of the weak projectivity of prime intervals in S is symmetric. The congruence relations on a distributive lattice S form a Boolean algebra if and only if S is discrete. These two results can be regarded as a contribution to the solution of the problem 72 of Birkhoff [loc.cit. p. 153]. An ideal of a lattice is called neutral if it is a congruence class under some lattice congruence relation. The correspondence between the congruence relations and neutral ideals of a discrete lattice S is one to one if and only if the following conditions are satisfied 1. To every prime interval p in S, there exists an ideal $I(p)$ such that $p \succeq q$ for every prime interval $q \subset I(p)$. 2. There exists a prime interval $p_1 \subset I(p)$ such that $p_1 \succeq p$. This is a contribution to the solution of the Problem 73 of Birkhoff [loc.cit. p. 161]. *M. Novotný* (Brno).

Maeda, Shûichirô. Dimension functions on certain general lattices. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 211–237.

The orthogonality-dimension theory for operator rings and the independence-dimension theory for continuous geometries (not assumed irreducible) are here included in a single abstraction; a vector-valued dimension is derived from an equivalence relation assumed given on a complete complemented lattice L. Modularity is not assumed but for each element in L there is postulated a non-empty set of particular complements (its independent complements) and the axioms on these complements imply a restricted modularity condition.

When each element in L happens to have a unique independent complement, the theory coincides with that given recently by L. H. Loomis [Mem. Amer. Math. Soc. no. **18** (1955); MR **17**, 514] for the orthocomplemented case. On the other hand, continuous geometries (and previously given generalizations) are special cases in which every complement is an independent one; then the equivalence relation abstracts the relation of perspectivity but without implying finiteness.

The symbol ∞ is used for all infinite dimensionalities and hence the results are more precise for finite elements. But infinite cardinals could have been used, as in Loomis, to obtain the more precise results for non-finite elements.

Von Neumann's independence theory is generalized as follows: Write $a \perp b$ to mean: a is included in some independent complement of b; let $a \oplus b$ denote $a \cup b$ and imply that $a \perp b$. The author assumes: (i) $a \perp b$ implies $b \perp a$, (ii) $(a \oplus b) \perp c$ implies $a \perp (b \cup c)$, (iii) $a \leqq c$ implies $c = a \oplus b$ for some b. If follows that $u(v \cup w) = v$ whenever $v \leqq u$ and $u \perp w$; this, in turn, implies that the b in (iii) must be of the form $a'c$ for some a' an independent complement of a (if independent complements are unique, (iii) is equivalent to the condition (M) of Loomis: $u(v \cup w) = v \cup uw$ whenever $v \leqq u$ and $v \perp w$, equivalently $a \oplus b = c$ whenever $a \leqq c$ and $b = a'c$ for arbitrary a' independent complement of a). Now L must be relatively complemented, z is in the centre Z (i.e. $z(a \cup b) = za \cup zb$ for all a, b) if and only if z has a unique complement (necessarily an independent one). Independence theory follows for finite sets of elements. The independence theory for non-finite sets of elements and the fact that Z is a complete Boolean sub-lattice of L is deduced with the use of an additional condition: (iv) if a_α is an increasing system for all $\alpha <$ some infinite ordinal and $b \perp a_\alpha$ for all α, then $b \perp \mathbf{U} a_\alpha$ (condition (iv) holds necessarily in the orthocomplemented case).

Suppose now that $\sim$ denotes a given equivalence relation in L. Call a, b incomparable if $a_1 \sim b_1$, $a_1 \leqq a$, $b_1 \leqq b$ together imply $a_1 = 0$; call b purely comparable to a if b_1, a are incomparable with $b_1 \leqq b$ only if $b_1 = 0$; call a invariant if b, a are incomparable whenever $ba = 0$. Assume: (i) $a \sim 0$ implies $a = 0$, (ii) $a \sim b_1 \oplus b_2$ implies $a = a_1 \oplus a_2$ for some $a_1 \sim b_1$, $a_2 \sim b_2$, (iii) a, b incomparable imply $a \perp b$, (iv) $x \leqq a \cup b$, with x, a incomparable and x, b incomparable together imply $x = 0$ (in the orthocomplemented case, (iv) follows from (iii), and (i), (ii), (iii) coincide with (A), (B), (D) respectively, in Loomis). For fixed a, let $a = a_1 < \cdots < a_\alpha < \cdots$ be an ascending transfinite sequence, maximal with the property that each a_α is purely comparable to a. Let $\bar{a} = \mathbf{U} a_\alpha$; then $\bar{a}$ is also purely comparable to a (for if $x \leqq \bar{a}$ with x, a incomparable, it follows that x, a_α are incomparable and $x \perp a_\alpha$ for every α, hence $x \perp \bar{a}$, $x \perp x$, $x = 0$). From (iv) it now follows that if b is purely comparable to a then $b \leqq \bar{a}$, hence $\bar{a}$ is invariant and every complement of $\bar{a}$ is independent of $\bar{a}$. In the orthocomplemented case of Loomis, this yields: $\bar{a}$ has a unique complement and so $\bar{a}$ is in the centre. For the general situation the author requires an additional postulate: (v) every invariant element is in the centre. Then the invariant elements form a complete Boolean sub-lattice of the centre, von Neumann's theory of central covers can be rewritten with $e(a) = \bar{a}$ denoting the least invariant element containing a (in place of the least central element containing a), minimal elements and finite elements can be defined and the reduction of L into sublattices of type I, II and III can be derived.

To complete the derivation of the dimension function the author assumes: (vi) $(a_\alpha) \perp$, $(b_\alpha) \perp$, $a_\alpha \sim b_\alpha$ for every α together imply $\mathbf{U} a_\alpha \sim \mathbf{U} b_\alpha$. This validates the Schroeder-Bernstein theorem. The author also assumes, where needed, that $a \cup b$ is finite if $a \perp b$ and a, b are both finite, but remarks that this necessarily holds even without requiring $a \perp b$ if: (vii) $a \sim b$ whenever a is perspective to b ((vi) and (vii) coincide with axioms (C), (D)′ respectively, in Loomis). *I. Halperin* (Kingston, Ont.).

Havel, Václav. Remark on the uniqueness of direct decompositions of elements in modular lattices of finite length. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5** (1955), 90–93. (Czech. Russian summary)

Let S be a modular lattice of finite length, and I the greatest element of S. Suppose that I has a direct decomposition into two factors. Then the following propositions are equivalent. (A) There is precisely one (to within order of the factors) direct decomposition of the element I into indecomposable factors. (B) Every element in S has at most one complement. (C) For every pair of direct decompositions $I = \mathbf{X}_{j=1}^{m} a_i = \mathbf{X}_{j=1}^{n} b_j$, the relations $a_i = \mathbf{V}_{j=1}^{n} (a_i \wedge b_j)$, $i = 1, 2, \cdots, m$, hold. (D) No sublattice of S is isomorphic with any of the four lattices whose diagrams are given. *M. Novotný* (Brno).

Tandai, Kwoichi. On certain pairs of mappings of modular lattices. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **5** (1955), 83–86.

Several theorems on endomorphisms of finite dimensional vector spaces as given in a paper by M. Hukahara [J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1954), 129–192; MR **16**, 992] are here recast into the language of lattice theory. The author's device is to look at ordered pairs of mappings (σ, λ) where σ (resp. λ) is a sup (resp. inf) preserving order-homomorphism of the modular lattice L (resp. M) into the modular lattice M (resp. L).

The author also studies the composition $\beta \cdot \alpha$ but does not observe that it is a closure operator on L.

R. M. Baer (Lafayette, Ind.).

Kudláček, Václav. Lattice-ordered groupoids. Časopis Pěst. Mat. **80** (1955), 44–50. (Czech. Russian and English summaries)

The lattices with a third operation (symbolized by $a \cdot b$) are studied. The following properties of this operation are considered:

(U)	$x \leqq y$ implies $a \cdot x \leqq a \cdot y$, $x \cdot a \leqq y \cdot a$,
(D1)	$a \cdot (x \cup y) = a \cdot x \cup a \cdot y$,
(D1′)	$a \cdot (x \cap y) = a \cdot x \cap a \cdot y$,
(D2)	$(x \cup y) \cdot a = x \cdot a \cup y \cdot a$,
(D2′)	$(x \cap y) \cdot a = x \cdot a \cap y \cdot a$,
(K)	$(a \cup b) \cdot (a \cap b) = a \cdot b$,
(H)	$a \cdot x \leqq a \cdot y$ if and only if $x \leqq y$, $x \cdot a \leqq y \cdot a$ if and only if $x \leqq y$,
(P)	$a \cdot x = a \cdot y$ implies $x = y$.

Some results: (D1) (D2) implies (U), while (U) does not imply (D1) (D2). The commutativity of the third operation together with (D1) (D1′) implies (K); (U) together with (K) implies the commutativity, but does not imply (D1) (D1′). (K) together with (P) implies the distributivity of the lattice. If the third operation is a quasi-group operation, then (D1) (D2) is equivalent to (H). Two sufficient conditions are given for a lattice whose third operation is a loop operation satisfying (H) to be simply ordered.

The reviewer's note: The definition of the third operation in the Example 1 requires the following correction: $c \cdot a = a \cdot b = a \cdot c = b$. The definition of the third operation in the Example 3 must be completed by $x \cdot 0 = x$ for every x. Theorem 4 requires the additional hypothesis that G has at least two elements. *M. Novotný* (Brno).

Jakubík, Ján. On metric lattices. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5** (1955), 140–143. (Slovak. Russian summary).

S being a lattice, let $|S|$ denote the set of all its elements

(without relations and operations). Let S, S' be two metric lattices [Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 76; MR 10, 673]. Suppose that S is constructed by the method of L. M. Kelly to a metric space, which satisfies Kelly's condition [Duke Math. J. 19 (1952), 661–669; MR 14, 494]. Consider the following propositions: (A) S and S', regarded as metric spaces, are identical. (B) $|S|=|S'|$ and there exist two lattices A, B and a one to one correspondence between $|S|$ and $|A|\times|B|$, which is an isomorphism between S and $A\times B$ and simultaneously an isomorphism between S' and $A\times B\subset\bar{A}$ is the dual of A). Then (A) is equivalent to (B). This is a solution of a problem proposed by L. M. Kelly. *M. Novotný* (Brno).

Watanabe, Sigekatu. Binary coding and isometric transformations in a finite metric Boolean algebra. Rep. Univ. Electro-Commun. 7 (1955), 17–40. (Japanese summary)

In this paper the subject of error-correcting codes is treated by the methods of abstract algebra using mappings groups, subgroups, maximal-subgroups, etc. In total some 27 theorems are given. The results are interesting to the mathematician, but do not seem to produce any essentially new information for the engineer.

R. W. Hamming (Murray Hill, N.J.).

See also: Aumann, p. 1180; Kolibiar, p. 1184; Kutyev, p. 1184; Linsman, p. 1258; Ivanov, p. 1259.

Rings, Fields, Algebras

Drazin, M. P. Algebraic and diagonable rings. Canad. J. Math. 8 (1956), 341–354.

An algebra R over a commutative ring (with unit) F is said to be algebraic if each r in R satisfies an equation $\alpha_n r^n+\alpha_{n-1}r^{n-1}+\cdots+r^m=0$, $m\geqq 1$, α_i in F. It is to be noted that the coefficient of the lowest power of r occurring is 1. The author first shows that if F is an integral domain, but not a field, and R is algebraic over F, then for each r in R, (*) $\alpha r^n=0$ for some (variable) integer n and $\alpha=\alpha(r)$ in F. Thus, e.g., the only algebras of characteristic 0 algebraic over the integers are nil algebras. Several weaker conditions are then given to ensure that (*) still holds for elements of R. An algebra R is next defined to be diagonable if for each r in R, $(\alpha_1 r+1)\cdots((\alpha_t r+1)=0$ with $\alpha_i=\alpha_i(r)$ in F. (The author assumes the α_i distinct, but this seems unnecessary.) It is easily seen that a diagonable algebra is strongly regular and thus is a subdirect sum of division rings [Arens and Kaplansky, Trans. Amer. Math. Soc. 63 (1948), 457–481; MR 10, 7]. Since a diagonable division algebra is commutative it follows that every diagonable algebra is commutative. The author then goes on to prove several highly abstract theorems concerning mappings on rings. From these several consequences are deduced; e.g. (i) A π-regular ring with no non-zero nilpotents is strongly regular. (ii) If a π-regular ring has all its nilpotents central, so does every homomorphic image. (iii) If R is algebraic over an algebraic extension field of a finite field and all its nilpotents are central, then R is commutative. The result (i) is already contained in Kaplansky, ibid. 68 (1950), 62–75 [MR 11, 317] and (iii) as well as theorem 5.7 are special cases of Lemma 3 of a paper by J. E. McLaughlin and the reviewer [Proc. Amer. Math. Soc. 4 (1953), 203–212; MR 14, 718]. (We should like to take this opportunity to point out that in the review of this last paper the statement of the main theorem is incomplete.)

In the last section the following conjecture from a letter of Herstein to the author is stated: If for each element r in a ring R there exists an element $a=a(r)$ in R such that $r-r^2a$ is central, then R is a subdirect sum of a commutative ring and division rings. Unfortunately, this is false as is shown by the two examples on pp. 207–208 of the paper by McLaughlin and the reviewer already cited. In both examples a non-commutative ring R with a single proper two-sided ideal N, lying in the centre, and such that $R-N$ is a field, is constructed. Thus if a is a representative of $(r+N)^{-1}$, $r-r^2a$ is in the centre but R is subdirectly irreducible. *A. Rosenberg* (Princeton, N.J.).

Amitsur, S. A. Radicals of polynomial rings. Canad. J. Math. 8 (1956), 355–361.

Let R be an arbitrary associative ring and denote by $J(R)$ the Jacobson radical of R [Amer. J. Math. 67 (1945), 300–320; MR 7, 2], and by $L(R)$ the lower radical of R [Baer, ibid. 65 (1943), 537–568; MR 5, 88]. The author then shows that if x is an indeterminate over R, $L(R[x])=L(R)[x]$ and $J(R[x])=N[x]$ where N is a nil-ideal in R containing the maximal locally nilpotent ideal of R. These results are then extended to polynomials in any number of variables. The only new feature that shows up is that for polynomials in an infinite number of variables the Jacobson radical is the maximal nil ideal. Finally the author indicates how to extend these results to a radical defined by a property π satisfying certain assumptions as in two of his papers [ibid. 76 (1954), 100–125, 126–136; MR 15, 499]. *A. Rosenberg* (Princeton, N.J.).

Goldie, A. W. Decompositions of semi-simple rings. J. London Math. Soc. 31 (1956), 40–48.

The author studies semi-simple rings having an irredundant representation as a subdirect sum of primitive rings. This is equivalent to having a set $\{P_\alpha\}$ of primitive ideals, $\alpha\in\Lambda_0$, with the property 1) $\bigcap_{\alpha\in\Lambda_0}P_\alpha=(0)$, 2) $\bigcap_{\alpha\in\Lambda_1}P_\alpha\neq(0)$ for every proper subset Λ_1 of Λ_0. Such a set of ideals is called a minimal decomposition set. If I is any non-zero ideal of R and if $I_r=\{x\in R|Ix=(0)\}$ and if R has a minimal decomposition set, then so do I and R/I_r. He finds conditions that a ring have such a minimal decomposition set. He defines a ring to be almost primitive if it is semisimple and if $I_r=(0)$ for any $I\neq(0)$ (i.e. almost primitive=prime semisimple). He shows that a semisimple ring with ascending chain condition on ideals is isomorphic to a subdirect sum of a finite number of almost primitive rings. *I. N. Herstein* (Philadelphia, Pa.).

Barnes, Wilfred E. Primal ideals and isolated components in noncommutative rings. Trans. Amer. Math. Soc. 82 (1956), 1–16.

L. Fuchs defined primal idesals in commutative rings and developed a theory of the representation of an ideal in a Noetherian ring as an intersection of primal ideals [Proc. Amer. Math. Soc. 1 (1950), 1–6; MR 11, 310]. C. W. Curtis has given an analogous theory for noncommutative rings [Amer. J. Math. 74 (1952), 687–700; MR 14, 127]. In the commutative case, however, the primal ideals of Fuchs are not necessarily primal in the sense of Curtis, although the converse is true. In this paper a definition of primal ideal in noncommutative rings is given which reduces to Fuchs' definition for commutative rings and to Curtis' for rings with unit element and ascending chain condition for ideals. (Definition: an ideal A of a ring R is

primal in case $\{x|x$ is "not right prime" to $A\}$ is an ideal of R, where x is not right prime to A in case $A \neq \{y|yRx \subseteq A\}$.) The Fuchs-Curtis decomposition theorems are obtained. A new definition of the right associated primes of an ideal is given, and the maximal such ideals determined. The remainder of the paper is devoted to a study of upper and lower right isolated B-components of an ideal A for B any divisor of A, where the methods of D. C. Murdoch [Canad. J. Math. **4** (1952), 43–57; MR **13**, 618] are employed. *R. D. Schafer* (Storrs, Conn.).

Steinfeld, Ottó. Bemerkung zu einer Arbeit von T. Szele. Acta Math. Acad. Sci. Hungar. **6** (1955), 479–484. (Russian summary)

A submodule Q of an (associative) ring R is a quasi-ideal of R if $(RQ) \cap (QR) \subseteq Q$. The author proves that if a proper quasi-ideal Q of R is not a one-sided ideal of R, then RQ and QR are proper left and right ideals of R, respectively. A construction is also given for obtaining a proper right (left) ideal of R from a given proper left (right) ideal. The paper of Szele quoted in the title is to be found in Acad. R. P. Române. Bul. Sti. A. **1** (1949), 783–789 [MR **12**, 237, 1002]. *S. A. Jennings.*

Kleinfeld, Erwin. Standard and accessible rings. Canad. J. Math. **8** (1956), 335–340.

A. A. Albert [Trans. Amer. Math. Soc. **64** (1948), 552–593; MR **10**, 349] introduced the study of standard rings, defined by the identities (1) $(wx, y, z)+(xz, y, w)+(wz, y, x)=0$ and (2) $(x, y, z)+(y, z, x)-(y, x, z)=0$, and proved that simple, finite-dimensional standard algebras are either Jordan or associative algebras. The author observes that (1) implies (3) $((w, x), y, z)=0$, and he calls a ring R accessible if (2) and (3) hold in R. Since a ring which is either commutative or associative is accessible, but (1) is not necessarily implied by the commutative law, every standard ring is accessible, but not conversely. The author proves that simple accessible rings, as well as primitive accessible rings, are either associative or commutative, thus generalizing Albert's results. He then uses the Brown extension [Proc. Amer. Math. Soc. **2** (1951), 114–117; MR **12**, 669] of the Jacobson radical to get a structure theory for semi-simple accessible rings. He further proves that if an accessible ring has no nilpotent ideals other than zero, then it is isomorphic to a subdirect sum of an associative ring and a commutative one. [Reviewer's remark: A footnote in the paper, referring to a result of the reviewer, is not quite correct. The reviewer has announced that flexible rings satisfying (3), and having no divisors of zero, are either associative or commutative. The flexible law is used in the reviewer's proof, and, indeed, there exist division rings satisfying (3) which are neither associative nor commutative.]

R. L. San Soucie (Eugene, Ore.).

Takasu, Tsurusaburo. Unified 'Schnitt'-like real representations of the three kinds of general binary complex numbers. Yokohama Math. J. **2** (1954), 127–136 (1955).

The author develops three kinds of "general binary complex numbers" as the discriminant of a certain quadratic is greater than, equal to, or less than zero. For these he develops in parallel form a "Schnitt-like" representation. *J. A. Ward* (Holloman, N.M.).

Takasu, Tsurusaburo. A unified theory of the three kinds of the general binary complex elements in the purely synthetic projective geometry. Yokohama Math. J. **2** (1954), 137–144 (1955).

The author develops three kinds of "general binary complex numbers" as the discriminant of a certain quadratic is greater than, equal to, or less than zero. For these he develops in parallel form a synthetic projective geometry. *J. A. Ward* (Holloman, N.M.).

Robinson, Abraham. Further remarks on ordered fields and definite functions. Math. Ann. **130** (1956), 405–409.

Verfasser gibt zunächst einen interessanten Beweis für den Artinschen Satz über die Darstellbarkeit total-positiver Elemente in formalreellen Körpern [und für dessen Verallgemeinerung durch den Verf. im Math. Ann. **130** (1955), 257–271; MR **17**, 822]. Es werden "pseudo-ideale" J im Körper definiert durch (1) mit a, b gehören auch $a+b$ und ab zu J, (2) alle Quadrate ausser 0 gehören zu J. Es ergibt sich, dass jedes pseudoideal Durchschnitt maximaler pseudoideale ist und dass die maximalen pseudoideale eineindeutig den totalen Ordnungen des Körpers entsprechen. Hieraus folgt unmittelbar der o.a. Artin-Robinsonsche Satz. Ferner gibt Verfasser eine Anwendung dieses Sätzes für Polynome, die auf algebraischen Mannigfaltigkeiten positiv-definit sind. *P. Lorenzen.*

Aumann, Georg. Bemerkung über Galois-Verbindungen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955**, 281–284 (1956).

If (P, Q) is a Galois-connection with mapping functions ϕ and ψ, then the author calls (P', Q') a contraction of (P, Q) if $\phi(P') \subseteq Q' \subseteq Q$ and $\psi(Q') \subseteq P' \subseteq P$. He proves that any Galois-connection is a contraction of a polarity [Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 54; MR **10**, 673]. *R. P. Dilworth* (Pasadena, Calif.).

Moisil, Gr. C. L'emploi des imaginaires de Galois dans la théorie des mécanismes automatiques. I. Sur les schémas à soupapes. Com. Acad. R. P. Romîne **4** (1954), 581–585. (Romanian. Russian and French summaries)

Si „ouvert" correspond avec 0, „fermé" avec 1, montage en série avec multiplication, montage parallel avec addition, l'ensemble (0, 0), (1, 0), (0, 1) et (1, 1) est isomorphe avec $GF(2^2)$. *O. Bottema* (Delft).

van Albada, P. J. Symmetric nonassociative algebras. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 265–268.

A not necessarily associative algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ is called symmetric if there is an involutory skew automorphism $T(a)=\bar{a}$ in $\mathfrak{A}$ which leaves only the elements of $\mathfrak{F}$ invariant. Every symmetric algebra is power-associative. The set $\mathfrak{A}'$ of ordered pairs (a, b), a, b in $\mathfrak{A}$ with sum $(a, b)+(c, d)=(a+c, b+d)$ and product $(a, b)\cdot(c, d)=(ac-\lambda\bar{d}b, da+b\bar{c})$ is also a symmetric algebra. Then $\mathfrak{A}'$ is commutative if and only if $\mathfrak{A}=\mathfrak{F}$ and in this case $\mathfrak{A}'$ is also associative. The algebra $\mathfrak{A}'$ is associative if and only if $\mathfrak{A}$ is commutative and associative; $\mathfrak{A}'$ is flexible if and only if $\mathfrak{A}$ is flexible; $\mathfrak{A}'$ is alternative if and only if $\mathfrak{A}$ is associative. *L. A. Kokoris* (New Haven, Conn.).

Kuroš, A. G. Nonassociative free sums of algebras. Mat. Sb. N.S. **37(79)** (1955), 251–264. (Russian)

The author exploits the methods of Witt [Math. Z.

58 (1953), 113–114; MR **15**, 5] to give simpler proofs of theorems on subalgebras of non-associative free algebras and isomorphic extensions for two free decompositions of an algebra [Kuroš, Mat. Sb. N.S. **20(62)** (1947), 239–262; MR **9**, 5]. The author also obtains Zukov's theorems on the free decomposition of algebras with a finite number of generators [ibid. **26(68)** (1950), 471–478; MR **12**, 238]. By modifying his previous definition of "free products" to "free sums", the author obtains the above mentioned results for algebras with identity as well as without identity. *L. J. Paige* (Los Angeles, Calif.).

Faddeev, D. K. On the concept of norm of a simple central algebra. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 662–663. (Russian)

Let k_0 be a field, k_1 a finite separable extension of k_0, $\mathfrak{a}$ a central simple algebra over k_1. M. Kneser [Arch. Math. **4** (1953), 97–99; MR **14**, 1058] defined the norm $N_{k_1/k_0}(\mathfrak{a})$; it is a central simple algebra over k_0. After observing (Th. 1) that $N_{k_2/k_0}(\mathfrak{a})=N_{k_1/k_0}(N_{k_2/k_1}(\mathfrak{a}))$, the author shows (Th. 2) that if, for any extension K of k_0, one writes $K\cdot k_1=\sum K_i$ (direct sum of fields) and $\mathfrak{a}_K=\sum \mathfrak{a}_i$ (direct sum, $\mathfrak{a}_i$ central simple over K_i), then $(N_{k_1/k_0}(\mathfrak{a}))_K\approx\prod N_{K_i/K}(\mathfrak{a}_i)$. Next, assuming that k_0 is a $\mathfrak{p}$-adic field and $\mathfrak{p}_1$ is a place of k_1 over $\mathfrak{p}$, he proves (Th. 3) that passing to the norm leaves invariant the invariant of $\mathfrak{a}$ at $\mathfrak{p}_1$, i.e. $(\mathfrak{a}/\mathfrak{p}_1)=(N_{k_1/k_0}(\mathfrak{a})/\mathfrak{p})$. This permits him to prove (Th. 4) that if k_0 is an algebraic number field of finite degree with place $\mathfrak{p}$ and $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ are the places of k_1 into which $\mathfrak{p}$ factors, then $(N_{k_1/k_0}(\mathfrak{a})/\mathfrak{p})=\sum(\mathfrak{a}/\mathfrak{p}_i)$. *E. R. Kolchin* (New York, N.Y.).

Rosenberg, Alex; and Zelinsky, Daniel. Cohomology of infinite algebras. Trans. Amer. Math. Soc. **82** (1956), 85–98.

This paper contains a variety of new results on the cohomological dimension of algebras. If A is an algebra over a commutative ring F (usually a field) the cohomological dimension of A, written dim A or, when the base ring must be mentioned, F-dim A, is defined as the largest non-negative integer n (or ∞, if there is no largest such n) for which the n-dimensional cohomology groups of A (in double A-modules) do not all vanish. [For the main known results concerning this notion, see Cartan and Eilenberg, "Homological algebra", Princeton, 1956; MR **17**, 1040; and the relevant papers of Eilenberg, Ikeda, Nagao, and Nakayama, cf. MR **16**, 214, 442, 993; **17**, 453 in lines 15 and 18 of the last cited review read "cohomological" for "global".] The main results obtained here are as follows. (I) If A is free over F and dim $A=0$, then A is finitely generated as an F-module. (II) Let A be an algebra over a commutative ring K, and assume that A is free over K. Let F be a subring of K. Then F-dim $K\leqq F$-dim $A\leq F$-dim $K+K$-dim A. (II) is used to establish the following facts for an extension field A of the field F, where $n\leqq\infty$ denotes the degree of transcendence of A over F. (1): dim $A\geqq n$; (2): if $n<\infty$, and A is finitely generated as a field over F, then dim $A=n$ or ∞ according to whether A is separably generated or not; (3): if $n<\infty$ and A is not finitely generated then dim $A>n$; (4): if A is countably (but not finitely) generated then dim $A=n+1$ or ∞ according to whether A is locally separably generated or not.

These results can be combined with (II) to give additional information on non-commutative algebras in a variety of cases. Finally, we mention a result of a somewhat different nature: let A be a locally finite semisimple algebra over a field F. Then, if A has countably infinite rank over F one has $H^1(A, A\otimes A)\neq 0$, while this 1-dimensional cohomology group vanishes when A has finite rank over F. *G. P. Hochschild.*

See also: Takasu, p. 1193; Numakura, p. 1223; Inaba, p. 1231; Inaba, p. 1231.

Groups, Generalized Groups

Pták, Vlastimil. On Frobenius' Theorem. Časopis Pěst. Mat. **78** (1953), 207–212. (Czech)

This paper is the result of a seminar on Zassenhaus' Lehrbuch der Gruppentheorie [Bd. I, Teubner, Leipzig-Berlin, 1937]. The author states that in this book at a number of places he would prefer a more leisurely exposition. As an instance he offers a detailed proof ab ovo of the theorem of Chap. I, § 10: Let G be a finite group of order N; let n be a natural number; let T be the class of all conjugates of an element $g\in G$ and h the number of elements in T. Let $A(T, n)$ be the number of different roots $x\in G$ of the equation $x^n=c$ where $c\in T$. Then $A(T, n)$ is divisible by the g.c.d. (hn, N). This is proved by induction with respect to N and the following three cases are considered separately: (1) g is not an element of the centre C of G; then $(hn, N)=A(T, n)$. (2) $g\in C$, but $g\neq 1$; the proof is based on the lemma: Let r be the order of c in G and p_i those prime factors of n which are not divisors of r; let e_i be the highest positive integer such that $p_i^{e_i}|n$. Let $\prod_i p_i^{e_i}=l$ and $n=kl$. Then $k|A(c, n)$. (3) $g=1$. *H. Schwerdtfeger* (Melbourne).

Berman, S. D. The number of irreducible representations of a finite group over an arbitrary field. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 767–769. (Russian)

Let G be a finite group and K a field. An element $g\in G$ is called K-regular if the order of g is not divisible by the characteristic of K. (If Char $(K)=0$ every element of G is K-regular.) Let n be the least common multiple of the orders of the K-regular elements in G; let ε be a primitive nth root of unity, and denote by I the set of integers i such that ε and ε^i are conjugate over K. Two K-regular elements $a, b\in G$ are called K-conjugate if there exist $c\in G$, $i\in I$ such that $b=c^{-1}a^ic$. Theorem: The number of classes of equivalent irreducible representation of G over K is equal to the number of classes of K-conjugate K-regular elements in G. The author proves this result by means of some results obtained earlier about p-adic character rings [same Dokl. (N.S.) **106** (1956), 583–586; MR **17**, 1052]. [Remark: For a proof of the above theorem see also E. Witt, J. Reine Angew. Math. **190** (1952), 231–245, p. 236, Satz 4; MR **14**, 845.] *P. Roquette.*

Šapiro, I. S. Expansion of a wave function in irreducible representations of the Lorentz group. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 647–649. (Russian)

The author gives an explicit representation as a direct integral of irreducible representations of the homogeneous Lorentz group of the unitary representation associated with a relativistic particle of zero spin and unit mass. This representation assigns to the homogeneous Lorentz transformation g the operator $\phi(p)\to\phi(g^{-1}p)$ on the Hilbert space of functions ϕ on the manifold $p_0^2-p_1^2-p_2^2-p_3^2=1$ with the norm $\|\phi\|^2=\int|\phi(p)|^2p_0^{-1}d_3p$. The direct integration takes place over a variable that is essentially the (scalar) value of the operator $\varepsilon_{\mu\nu\lambda\lambda}M_{\mu\nu}M_{\lambda\lambda}$

in the particular irreducible representation, where $M_{\mu\nu}=-i(p_\mu\partial/\partial q_\nu-p_\nu\partial/\partial q_\mu)$; $\mu=1, 2, 3, 4$; $p_4=ip_0$.

I. E. Segal (Chicago, Ill.).

Littlewood, D. E. The characters and representations of imprimitive groups. Proc. London Math. Soc. (3) **6** (1956), 251–266.

In this paper the author obtains the representation theory of the group H of order $(r!)^s s!$ which is the normalizer of the direct product of s factors S_r on distinct sets of symbols in S_{rs} [Robinson, Canad. J. Math. **1** (1949), 166–175; **2** (1950), 334–343; MR **10**, 504; **12**, 74]. He uses the language of Schur functions and assumes a familiarity on the part of the reader with the necessary connection with the symmetric group and its induced representations. The chief contribution of the paper is a careful analysis of the structure of H, completing previous discussions. While it is true that H can be considered as a generalization of the hyperoctahedral group ($r=2$) of order $2^s s!$, another natural generalization is the group $S(s, p)$ of order $p^s s!$ whose representation theory has also been obtained [Osima, ibid. **6** (1954), 511–521, §3; MR **16**, 566].

G. de B. Robinson (Toronto, Ont.).

Frucht, Robert. Zur Darstellung endlicher Abelscher Gruppen durch Kollineationen. Math. Z. **63** (1955), 145–155.

The representations of finite abelian groups by collineations were studied by the author in an earlier paper [J. Reine Angew. Math. **166** (1931), 16–29]. He then used Schur's general theory of representations of finite groups by collineations [ibid. **127** (1904), 20–50; **132** (1907), 85–137]. In the present paper, the author treats the same problem directly. He also determines the representations of the dihedral groups by collineations. *R. Brauer.*

Osima, Masaru. Notes on blocks of group characters. Math. J. Okayama Univ. **4** (1955), 175–188.

A number of results concerning blocks of group characters have been given by the reviewer [Proc. Nat. Acad. U.S.A. **30** (1944), 109–114; **32** (1946), 182–186, 215–219; MR **6**, 34; **8**, 14, 131]. A general outline of the method was indicated, but no detailed proofs were given. The author proves the main results of the first two papers. In the proof of the central theorem, he follows the outline mentioned above. However, the reviewer feels that the proof as given is hardly complete. Since it was the purpose of some of the other results, to lead to a proof of this theorem, it is not difficult to close the gap. The following new results may be mentioned. An explicit formula is given for the idempotent of the group algebra belonging to a given block. This result is used to give very simple proofs for some of the lemmas needed for the proof of the main result, simpler in fact than the corresponding proofs of the reviewer. Further, it is shown that if $\chi_1, \chi_2, \cdots, \chi_l$ is a set of ordinary irreducible characters of the group such that $\sum_{i=1}^{l}\chi_i(R)\chi_i(S)$ for every p-regular element R and every p-singular element S of the group, then $\chi_1, \cdots, \chi_l$ is a collection of blocks. [The details of the proofs of the reviewer for the results mentioned have appeared recently in Math. Z. **63** (1956), 406–444].

R. Brauer (Cambridge, Mass.).

Ibrahim, E. M. Tables for the plethysm of S-function of degree 10 and 12. Proc. Math. Phys. Soc. Egypt **5** (1954), no. 2, 85–86 (2 plates) (1955).

The tables given here record the reduction of $\{\lambda\}\otimes\{\mu\}$, where (λ) is a partition of n and (μ) is a partition of m, for $nm=10$ or 12. *G. de B. Robinson* (Toronto, Ont.).

Ibrahim, E. M. On D. E. Littlewood's algebra of S-function. Proc. Amer. Math. Soc. **7** (1956), 199–202.

Once again the problem of the reduction of $\{\lambda\}\times\{\mu\}$ in the general case is considered. If we call $\{\lambda_1+\mu_1\}$ the principal part of the reduction of the product $\{\lambda\}\{\mu\}$, then the author shows that the principal parts of the products of terms in the expansion $(\{\lambda\}\otimes\{u\})(\{\mu\}\otimes\{v\})$ appear as terms in the expansion $\{\lambda_1+\mu_1\}\otimes\{w\}$ whenever $x_\varrho^u x_\varrho^v=x_\varrho^w$ for all ϱ of S_n. This does not determine the coefficients but supplements a method of D. E. Littlewood for obtaining the reduction in question. Tables giving the reduction for $nm\leqq 18$ have been worked out [cf. the preceding review].

G. de B. Robinson (Toronto, Ont.).

Ibrahim, E. M. Some subgroups of the orthogonal and symplectic groups. Proc. Math. Phys. Soc. Egypt **5** (1954), no. 2, 9–15 (1955).

Pursuing the ideas of D. E. Littlewood the author uses previously calculated tables of $\{\lambda\}\otimes\{\mu\}$ to study subgroups of the orthogonal and symplectic groups through their invariant forms. *G. de B. Robinson.*

Vosper, A. G. Addendum to "The critical pairs of subsets of a group of prime order". J. London Math. Soc. **31** (1956), 280–282.

This note contains a simpler proof of the results of the quoted paper [same J. **31** (1956), 200–205; MR **17**, 1182].

P. Scherk (Philadelphia, Pa.).

Tobin, Seán. On a theorem of Baer and Higman. Canad. J. Math. **8** (1956), 263–270.

The Frattini series of a group is formed by iterating the process of constructing the Frattini subgroup just as the derived series is formed by iterating the process of constructing the derived group. Let a finitely generated group G have exponent 4, and let the Frattini series reach the identity subgroup in precisely j steps. Then the class of the Frattini subgroup is on the interval $[2^{j-2}, 5^{j-2}]$. Let a finitely generated group G have prime exponent p, and let the derived series reach the identity in precisely d steps. Then the class of G is on the interval $[2^{d-1}, p^{d-1}]$. The lower bounds are due to Hall [Proc. London Math. Soc. (2) **36** (1933), 29–95], but the upper bounds are established in this paper. For the groups in question, the author is able to show that certain deep members of the lower central series lie in comparatively shallow members of the derived series. He is able to show that if a member of the lower central series of a finitely generated group of prime-power exponent lies in the first significant member of the lower central series of the derived group, then each member of the latter series includes members of the former. *F. Haimo* (St. Louis, Mo.).

Cohn, P. M. The complement of a finitely generated direct summand of an abelian group. Proc. Amer. Math. Soc. **7** (1956), 520–521.

The author proves that if a finitely generated abelian group is a direct summand of an abelian group, then the complementary summand is unique to within an isomorphism. This answers affirmatively problem III, p. 13, of Kaplansky "Infinite abelian groups" [Univ. of Michigan Press, 1954; MR **16**, 444]. This result has been found independently by E. A. Walker. *F. Haimo.*

Charles, B. Etude des groupes abéliens primaires de type $\leq\omega$. Ann. Univ. Sarav. **4** (1955), 184–199 (1956).

Let G be a primary abelian group, P the subgroup of elements of order p, P_α the subgroup of P consisting of elements with height at least α. In this way we associate with G a vector space over GF(p) with a well-ordered decreasing sequence of subspaces. The author studies this type of structure and then applies the results to obtain the theory of groups with no elements of infinite height.

I. Kaplansky (Madison, Wis.).

Nefed'ev, G. N. Periodic abelian extensions of a group of type p^∞. Ural. Politehn. Inst. Trudy. **51** (1954), 92–106. (Russian)

If the quotient-group G/A is the direct product of subgroups $\bar{H}_1$ and $\bar{H}_2$ and if G is the direct product of subgroups H_1 and H, where H_1 is isomorphic to $\bar{H}_1$ and $H\supset A$, then H is called a direct factor of the extension G of A by G/A. Let the periodic abelian group $\bar{G}$ be the direct product of two subgroups of which one, $\bar{H}$, does not contain elements of order a power of the prime p which have finite height in $\bar{H}$; if the extension of a complete abelian finitely-layered p-group A by $\bar{H}$, which is induced by the extension G of A by $\bar{G}$, is a central extension, then $\bar{H}$ is a direct factor of the extension G. Let the periodic reduced abelian group $\bar{G}$ be the direct product of a Sylow p-subgroup $\bar{H}$ and a non-trivial subgroup $\bar{F}$; if there is associated with at least one element of $\bar{F}$ a non-identity automorphism of a group A of type p^∞, then the extension G of A by $\bar{G}$ is a normal extension and $G=H\times(AF)$, where H and F are isomorphic respectively to $\bar{H}$ and $\bar{F}$. Details are provided for the structure of an extension of a group of type p^∞ by a direct product of cyclic groups of orders which are powers of p. *R. A. Good* (College Park, Md.).

Solian, Alexandru. Sur la notion de "n-complet" dans les groupes. Acad. Repub. Pop. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 255–272. (Romanian. Russian and French summaries)

Definitions: A group G is n-complete if nth powers of its elements form the whole group; n-quasicomplete if $K_n(G)$, the subgroup generated by nth powers, is the whole group; n-metacomplete if there is a chain of subgroups leading from G to the identity E, each normal in the preceding, with n-complete quotient groups. If a representative a_λ of each conjugate class excluding E is expressed as a product of n factors in G, $a_\lambda=b_{\lambda_1}b_{\lambda_2}\cdots b_{\lambda_n}$, the elements $t^{-1}b_{\lambda_i}b_{\lambda_j}{}^{-1}t$ ($i,j=1,\cdots,n$; $t\in G$) generate an n-canonical normal subgroup. The n-metacomplete radical C_n is an n-metacomplete normal subgroup which contains all n-metacomplete normal subgroups.

Theorems: A homomorph of an n-complete group is n-complete. A homomorph of an arbitrary group is n-complete if and only if the kernel of the homomorphism contains an n-canonical normal subgroup. A necessary and sufficient condition that an abelian group G should possess a non-trivial n-complete homomorph is: the orders of the elements of G either include a number having a prime factor not contained in n, or include ∞, or are finite but form a set not bounded above. n-quasicomplete with abelian implies n-complete. $K_n(G)\cup N=G$ if N is an n-canonical normal subgroup. n-metacomplete implies n-quasicomplete. Homomorphs and unions of n-metacomplete groups are n-metacomplete. C_n exists if G satisfies the maximal condition on normal chains. G/C_n contains no non-trivial n-metacomplete normal subgroup. The whole theory can be generalised on the following lines: replace n-completeness by the property that any element a can be decomposed into $a=x_1x_2\cdots x_n$ where (instead of $x_ix_{i+1}{}^{-1}=e$, the identity) $\phi_i(x_1,x_2,\cdots,x_n)=e$ ($i=1,\cdots,n-1$); the ϕ's here are $n-1$ fixed but arbitrary functions of a rather general kind based on the group operation.

Analogies with the theories of abelian homomorphs of groups, metabelian solvable and nilpotent groups are suggested. *I. M. H. Etherington* (Edinburgh).

Neumann, Hanna. On varieties of groups and their associated near-rings. Math. Z. **65** (1956), 36–69.

A variety of groups is determined by the set of laws its members satisfy, and hence by a fully invariant subgroup V of a free group F of countable rank. The first part of this paper is concerned with the extent to which the variety is determined by its n generator groups for finite n. If F_n is a free factor of F of rank n, and $V^{(n)}$ the least fully invariant subgroup of F containing $F_n\cap V$, then the groups $V^{(1)}, V^{(2)}, \cdots$ form an ascending chain converging to V. Similarly, if W^n is the product of all fully invariant subgroups W of F such that $F_n\cap W=F_n\cap V$, then W^n is fully invariant, and the groups $W^1, W^2, \cdots$ form a descending chain converging to V. $V^{(n)}$ and W^n define respectively the greatest and the least variety having the same n generator groups as that defined by V. The second derived group provides an example with $V^{(1)}\neq V^{(2)}\neq V^{(3)}\neq V^{(4)}=V$, and the subgroup generated by the cubes of elements of F an example with $W^1\neq W^2\neq W^3=V$, but it is not known whether either chain can be genuinely infinite, i.e., whether there is a variety not determined by its n generator groups for any n.

In the second part, the endomorphisms of a relatively free group F are made into a near-ring Γ. Multiplication is as usual, but to define addition, one takes once for all a fixed free basis E, and writes $e(\alpha+\beta)=e\alpha\cdot e\beta$ for e in E, α,β in Γ. This is possible, because each mapping of E into F determines just one endomorphism. Of special importance is the subring Γ_m of Γ, consisting of those endomorphisms which map all but a finite set of elements of E on 1. Γ_m is additively generated by the endomorphisms ε such that there is just one e in E for which $e\varepsilon\neq 1$, and for this e, $e\varepsilon$ is in E. These elements have the special property that multiplication by them is distributive on both sides. This enables the author to reestablish the classical connection between homomorphisms and ideals for the near-rings Γ_m, and to show that the essentially only homomorphisms of Γ_m are the obvious ones onto the near-rings associated with F/V for fully invariant subgroups V of F. Thus the structure of the set of all fully invariant subgroups of F is mirrored in the ideal-theoretic structure of Γ_m. This correspondence shows itself most clearly when F is of infinite rank; then the sum and intersection of ideals correspond to the join and intersection of fully invariant subgroups, and the multiplication of ideals to the operation of substituting the elements of one fully invariant subgroup into the words defining another. If F is of finite rank, the situation is more complicated, and ideal multiplication need not be associative. Particular attention is given to the case when F is absolutely free and of infinite rank. Here right multiplication is an isoendomorphism of the lattice of ideals; and every ideal is a product of indecomposable ideals.

It should be remarked that the statement in the penultimate paragraph described as a conjecture of the reviewer is in fact an unpublished theorem.

Graham Higman (Oxford).

Kutyev, K. M. SL-**isomorphisms of partially ordered locally nilpotent groups.** Uspehi. Mat. Nauk (N.S.) **11** (1956), no. 2(68), 193–198. (Russian)

An SL-isomorphism of a partially ordered locally nilpotent torsion-free group G is an isomorphism mapping the lattice of all semigroups in G (including the empty semigroup) onto the corresponding lattice of a group G^*. A semigroup in G not containing the inverses of its elements is called pure. Various properties preserved by SL-isomorphisms are studied, culminating in the principal result that under an SL-isomorphism of G the image of a pure isolated invariant semigroup with identity is a semigroup in G^* with the same properties. *R. A. Good.*

Klein, Micha. The polyedromodular quotient of the 5-th order compared with the groups associated with the rotational symmetry of the icosahedron and the rotational symmetry of the 5 cell. J. Osaka Inst. Sci. Tech. **4** (1952), 1–17.

Der Zusammenhang der Ikosaerdegruppe und der Gruppe des Fünfzells im vierdimensionalen Raum wird untersucht. *J. J. Burckhardt* (Zürich).

Tetsuya, Kazutoshi; Hashimoto, Takao; Akazawa, Tadao; Shibata, Ryōichi; Inui, Tadashi; and Tamura, Takayuki. All semigroups of order at most 5. J. Gakugei Tokushima Univ. Nat. Sci. Math. **6** (1955), 19–39. Errata on loose, unpaginated sheet.

All types of semigroups of order 5, obtained without the assistance of a computing machine, are listed in abbreviated form. The 1160 types, no two of which are isomorphic or anti-isomorphic, are classified according to greatest homomorphic semilattice S. Of these, the 165 semilattice indecomposable types, i.e. those with $S=1$, are sub-classified. Semigroups of order less than 5 are similarly listed, together with their automorphisms and "translations". The authors give 1-1 correspondences (a) to Tamura's previous list of semigroups of order 4 [J. Gakugei Tokushima Univ. Math. **5** (1954), 17–27; MR **16**, 1085], and (b) to a list of semigroups of order 5 obtained by T. S. Morzkin and the reviewer, using the SWAC [Bull. Amer. Math. Soc. **62** (1956), 14]. *J. L. Selfridge* (Los Angeles, Calif.).

Tamura, Takayuki. One-sided bases and translations of a semigroup. Math. Japon. **3** (1955), 137–141.

This paper purports to be one of a pair about translations of a semigroup S, the other to be published in Kōdai Math. Sem. Rep. A translation is a mapping $\phi(z)$ such that $\phi(xy)=x\phi(y)$. Among all subsets X which satisfy the relation $SX\cup X=S$ those that are minimal are called "right bases." There exist such subsets X in every semigroup, but there may be none that are minimal. This is shown by the semigroup of the positive integers where $xy=\min(x, y)$. All right bases of a semigroup have the same cardinal. A translation $\phi(z)$ is uniquely determined by its values on a right base. *H. Campaigne.*

Miller, D. D.; and Clifford, A. H. Regular $\mathcal{D}$-classes in semigroups. Trans. Amer. Math. Soc. **82** (1956), 270–280.

Green [Ann. of Math. (2) **54** (1951), 163–172; MR **13**, 100] has introduced the following equivalence relations in an arbitrary semigroup S: $a\mathcal{L}b$ ($a\mathcal{R}b$) if a, b generate the same left (right) ideal, $\mathcal{D}=\mathcal{L}\times\mathcal{R}$ and $\mathcal{H}=\mathcal{L}\cap\mathcal{R}$. ($\mathcal{D}$ is an equivalence relation since $\mathcal{L}$ and $\mathcal{R}$ are permutable.) For any $a\in S$ Let R_a, L_a, D_a, H_a denote the respective $\mathcal{R}$-, $\mathcal{L}$-, $\mathcal{D}$-, and $\mathcal{H}$- equivalence classes to which a belongs.

In the first part of this paper a number of new and deeper results concerning the multiplication of such classes is obtained. An element $a\in S$ is called regular if $a\in aSa$. An element a' is called an inverse of a if $aa'a=a$ and $a'aa'=a'$. If a $\mathcal{D}$-class D contains a regular element, then every element of D is regular. In this case every $\mathcal{L}$-class and every $\mathcal{R}$-class in D contains at least one idempotent. Every inverse of a regular element a lies in D_a; an $\mathcal{R}$-class H_b contains an inverse of a if and only if both of the $\mathcal{R}$-classes $R_a\cap L_b$ and $R_b\cap L_a$ contain idempotent elements; no $\mathcal{R}$-class contains more than one inverse of a. Any inverse of an idempotent need not be itself idempotent, but is always a product of two idempotents. The maximal subgroups of a regular $\mathcal{D}$-class D are all isomorphic with one another.

Some known results are obtained from a unified point of view.

In the second part of the paper it is shown: to every regular $\mathcal{D}$-class D of S we can associate (uniquely to within isomorphism) a regular matrix semigroup $\mathfrak{M}$ [in the sense of Rees, Proc. Cambridge Philos. Soc. **36** (1940), 387–400; MR **2**, 127] in such a manner that D is in a certain sense partially isomorphic with $\mathfrak{M}$. If S is completely simple the construction of $\mathfrak{M}$ reduces to a result of Rees-Suschkewitsch (loc.cit. above). *S. Schwarz.*

Iséki, Kiyoshi. Contribution to the theory of semigroups. I. Proc. Japan Acad. **32** (1956), 174–175.

A semigroup S is called a torsion semigroup (in Russian terminology: periodic semigroup) if every element $\in S$ has a finite order. S is said to be strongly reversible if for any two a, $b\in S$ there exist three positive integers r, s, t such that $(ab)^r=a^sb^t=b^ta^s$. Any strongly reversible torsion semigroup is the sum of disjoint semigroups each containing exactly one idempotent. This is a generalisation of a result of the reviewer [Czechoslovak Math. J. **3**(**78**) (1953), 7–21; MR **15**, 850]. A new proof of the following known result is given: In a torsion semigroup containing only one idempotent e we have: $eSe=SeS=$maximal subgroup of $S=$minimal ideal of S. *S. Schwarz.*

Kolibiar, Milan. On permutable relations. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5** (1955), 137–139. (Slovak. Russian summary)

The author studies the class of all binary relations on a fixed set; the order and the multiplication of the relations are defined in the usual manner [Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 209; MR **10**, 673]. The results: For the symmetric relations θ, θ', the inequality $\theta\theta'\leqq\theta'\theta$ implies $\theta\theta'=\theta'\theta$. As a corollary it follows that a congruence relation θ on a loop, for which $ux\equiv x$ (θ) implies $u\equiv 1$ (θ), is permutable with any congruence relation. This is a solution of the Problem 32 of Birkhoff [loc.cit. p. 86]. As the author remarks in a note added in proof, the solution of this problem was published earlier by Trevisan [Rend. Sem. Mat. Univ. Padova **22** (1953), 11–22; MR **15**, 681].

The reviewer's note: The Lemmas 2.1, 2.2, 2.3 are contained in the Theorems 1 and 10 of the Chapter XIII of Birkhoff's book. *M. Novotný* (Brno).

See also: Kudláček, p. 1178; Venkov, p. 1186.

THEORY OF NUMBERS

Eda, Yoshikazu. On the Burg's numbers. Sci. Rep. Kanazawa Univ. 3 (1955), 213–215.

The author studies the structure of those positive integers N having the property that the integer obtained by reversing the digits of N, relative to a given base g, is divisible by N. *H. N. Shapiro* (New York, N.Y.).

Lohne, Johannes. Graphical illustration of summation formulae. Nordisk Mat. Tidskr. 4 (1956), 85–86, 120. (Norwegian. English summary)

The well-known formulae for $\sum_{x=1}^{n} x^m$ ($m=1, 2, 3$) are illustrated graphically. *Author's summary.*

Campbell, J. G. On the representation of powers of primes by polynomials with integral zeros. Amer. Math. Monthly 63 (1956), 413.

Sierpiński, W. Sur les décompositions de nombres rationnels en fractions primaires. Mathesis 65 (1956), 16–32.

Let $a, b, x_1, x_2, \cdots$ be positive integers. Denote by $f(a, b)$ the smallest value of s for which

$$\frac{a}{b}=\pm\frac{1}{x_1}\pm\frac{1}{x_2}\pm\cdots\pm\frac{1}{x_s}$$

is solvable, and by $g(a, b)$ the smallest integer s' for which

$$\frac{a}{b}=\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_r}$$

is solvable. The author proves (among other results) that $f(18, 23)=4$ and that the set of rational numbers a/b with $f(a, b)\leqq s$ is nowhere dense for every s. Schinzel conjectured that for every a there exists a constant $c=c(a)$ so that for $b>c$, $f(a, b)\leqq 3$.

The reviewer and Straus conjectured some years ago that $g(4, n)\leqq 3$ for every $n>1$. Schinzel conjectures that for every a there exists a $c'=c'(a)$ so that for $b<c'$, $f(a, b)\leqq 3$. (The author first conjectured $g(5, n)\leqq 3$ and proved it for $1<n\leqq 1000$). *P. Erdős* (Haifa).

Duparc, H. J. A.; and Peremans, W. On theorems of Wolstenholme and Leudesdorf, additional note. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 67–69.

This note gives an extension of the second theorem of the authors' previous paper [same Proc. 58 (1955), 459–465; MR 17, 348]. There is also a discussion of the related results and methods of N. Eljoseph [Riveon Lematematika 4 (1950), 9–15, 59–61; MR 13 437]. *D. H. Lehmer.*

★**Mordell, L. J. The diophantine equation $x^3+y^3+z^3+kxyz=0$.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 67–76. Georges Thone, Liège; Masson and Cie, Paris, 1956.

The equation of the title is shown to have, for $k=1$, no solutions in relatively prime integers (x, y, z) other than the trivial ones $(1, 1, -1)$, $(1, -1, -1)$, $(1, -1, 0)$ and the obvious permutation thereof. For $k=-5$ there are no solutions other than $(1, -1, 0)$ $(1, 1, 2)$, $(-1, -1, -2)$ and their permutations. These results apparently fill a hiatus left by A. Hurwitz [Vierteljschr. Naturforsch. Ges. Zürich 62 (1917), 207–229] in his study of homogeneous cubic equations. *I. Niven* (Eugene, Ore.).

Ljunggren, Wilhelm. New theorems concerning the diophantine equation $Cx^2+D=y^n$. Norske Vid. Selsk. Forh., Trondheim 29 (1956), 1–4.

Let C and D be positive integers such that CD is square-free, $CD\not\equiv 7 \pmod 8$, D of the form $1+4^m d$ with d odd. Assume further that either $Cd\equiv 5 \pmod 8$ or $C=1$ and $d\equiv 3 \pmod 8$. Then the equation $Cx^2+D=y^p$ has only a finite number of solutions (x, y, p) in positive integers x, y and primes $p>3$ such that p does not divide the class number of the field $R(\sqrt{-CD})$. This extends an earlier result by the writer [Norske Vid. Selsk. Forh., Trondheim 18 (1945), no. 32, 125–128; MR 8, 136]. It is also proved that $x^2+21=y^n$ and $x+13=y^n$ are impossible in positive integers $x, y, n>2$ apart from the solution $x=70$, $y=17$, $n=3$ of the second equation. *I. Niven.*

Selfridge, J. L.; Nicol, C. A.; and Vandiver, H. S. On Diophantine equations which have no solutions. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 264–266.

The authors consider the number of solutions of the congruence $1+ax^e\equiv by^l \pmod p$, where $p-1=lc$. A set of tables giving the number of solutions of the congruence was prepared on the SWAC for $l=5, 7, 11, 13$ and 17 for values of p ranging from $p<1024$ for $l=5$ and 7 to $p<512$ for $l=17$.

Application is made of the non-existence of solutions of the congruence to the unsolvability of the corresponding Diophantine equation.

It is stated without proof that it is possible to show that there are at least two congruences for each c and l which have no solutions modulo $p=cl+1$. *D. H. Lehmer.*

Ankeny, N. C. The law of quadratic reciprocity. Norske Vid. Selsk. Forh., Trondheim 28 (1955), 145–149 (1956).

The author gives a proof of the law of quadratic reciprocity using cyclotomic fields. He makes no use of the Gauss lemmas. *P. Erdős.*

See also: Pták, p. 1181.

Analytic Number Theory

Kanold, Hans-Joachim. Über einen Satz von L. E. Dickson. Math. Ann. 131 (1956), 167–179.

Let $\sigma(n)$ denote the sum of the positive divisors of the integer n, and $V(n)$ the number of distinct prime factors of n. It is proved that for any fixed integers s, k and a, there is only a finite number of integers n such that $\sigma(n)=sn$, $V(n)\leqq k$ and $2^a\|n$. This generalizes a result of L. E. Dickson [Amer. J. Math. 35 (1913), 413–422, 423–426] that there is only a finite number of odd perfect numbers with a given number of distinct prime factors. It is also established that there is only a finite number of integers n such that $\sigma(n)=sn$ and $V(n)\leqq k$ in case s is a power of 3 or a power of 5. *I. Niven* (Eugene, Ore.).

Mitrinovitch, D. S. Su un determinante e sui numeri di Stirling che vi si collegano. Boll. Un. Mat. Ital. (3) 11 (1956), 93–96.

Let $r_1, r_2, \cdots, r_n$ be distinct numbers and let $D_{n,k}$ denote the determinant whose ith row is

$$1\binom{r_i}{1}\cdots\binom{r_i}{k-1}\binom{r_i}{k+1}\cdots\binom{r_i}{n}$$

(the column of binomial coefficients $\binom{r_i}{k}$ being omitted). The author finds that

$$D_{n,k}=\frac{V_n(\lambda_{k0}\sigma_{n-k}+\lambda_{k1}\sigma_{n-k-1}+\cdots+\lambda_{kn-k})}{[1!2!\cdots(k-1)!(k+1)!\cdots n!]}.$$

Here V_n is the Vandermonde determinant of the r_i, the σ_j are the elementary symmetric functions of the r_i, and the coefficients λ_{pq} satisfy the recurrence relation

$$\lambda_{pq}=$$

$$\lambda_{p+1,q-1}S_{p+1}{}^{p}-\lambda_{p+2,q-2}S_{p+2}{}^{p}+\cdots+(-1)^{q+1}\lambda_{p+q,0}S_{p+q}{}^{p}$$

in which the $S_n{}^m$ are the Stirling numbers of the first kind, defined by the identity $r(r-1)\cdots(r-k+1)\equiv S_k{}^1r+S_k{}^2r^2+\cdots+S_k{}^{k-1}r^{k-1}+S_k{}^kr^k$. The proof is not given. Reference is made to a note by the reviewer [J. London Math. Soc. **24** (1949), 144–145; MR **10**, 671] where a particular case of the above determinant is treated by a different method. The case of $D_{n,n}$ can also be found in a paper by Stern, J. Reine Angew. Math. **66** (1866), 285–288. *K. A. Hirsch* (London).

Venkov, B. A. On an integral invariant of the group of unimodular linear substitutions. Leningrad. Gos. Univ. Uč. Zap. **144**. Ser. Mat. Nauk **23** (1952), 3–25. (Russian)

For positive integral n, let $\mathfrak{R}$ be the $\nu=n(n+1)/2$ dimensional space of all real quadratic forms

$$f(X)=\sum_{p,q=1}^{n}a_{pq}x_px_q=X'AX,\quad X'=(x_1,\cdots,x_n),$$

$$A=(a_{pq})=A',$$

(the dash denotes the transposed matrix), and let $\mathfrak{P}$ be the cone of all positive-definite forms. Denote by Γ the group of all integral unimodular matrices S. Every transformation $X=SX_1$ changes f into an equivalent form f_1 of matrix $S'AS$. In each class of equivalent forms there is one and in general only one representative which is reduced in the sense of Hermite and Minkowski. Denote by $\mathfrak{G}\subset\mathfrak{R}$ the set of all such reduced points. Minkowski proved [J. Reine Angew. Math. **129** (1905), 220–274= Ges. Abh., Bd. II, Teubner, Leipzig-Berlin, 1911, pp. 53–100] that $\mathfrak{G}$ is a convex conical polyeder bounded by finitely many planes. If

$$dA=\prod_{1\le p<q\le n}da_{pq}$$

defines the volume element in $\mathfrak{R}$, Minkowski further showed that the volume v_n of the set of all reduced points of determinant $\det A\le 1$ equals

$$v_n=\frac{2}{n+1}\pi^{-n(n+1)/4}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{2}\right)\cdots\Gamma\left(\frac{n}{2}\right)\zeta(2)\zeta(3)\cdots\zeta(n).$$

A second proof was given by C. L. Siegel [Trans. Amer. Math. Soc. **39** (1936), 209–218, pp. 210–211], who also obtained a third proof as a by-product in a more recent paper [Ann. of Math. (2) **46** (1945), 340–347; MR **6**, 257] which is not mentioned by the author. The author derives Minkowski's formula by the following analytical considerations.

For complex $s=\sigma+ti$, let

$$\Phi(s)=\pi^{-\frac{1}{2}ns-\frac{1}{4}n(n-1)}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\cdots\Gamma\left(\frac{s+n-1}{2}\right).$$

If M is any integral matrix of determinant $\det M=\mp\Delta$, $\Delta\ge 1$, and if $\sigma(M'AM)$ denotes the trace of $M'AM$, then

$$\Delta^{-(n+s-1)}\Phi(s)=\int_{\mathfrak{P}}e^{-\sigma(M'AM)}(\det A)^{\frac{1}{2}s-\frac{1}{2}}dA.$$

Write now $\mathfrak{P}$ as the union of all polyhedra $\mathfrak{G}$, $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\cdots$ equivalent to $\mathfrak{G}$ under Γ, and transform the respective integrals into such over $\mathfrak{G}$. Then

$$2\Delta^{-(n+s-1)}\Phi(s)=\int_{\mathfrak{G}}(\det A)^{\frac{1}{2}s-\frac{1}{2}}\sum_{S}e^{-\sigma(M'S'ASM)}dA,$$

where S runs over the elements of Γ. On summing here first over all classes of matrices SM, $S\in\Gamma$, and then over the determinants $\Delta=1, 2, 3, \cdots$, one obtains the identity $2\Phi(s)\zeta(s)\zeta(s+1)\cdots\zeta(s+n-1)=$

$$\int_{\mathfrak{G}}(\det A)^{\frac{1}{2}s-\frac{1}{2}}\sum_{\det M\neq 0}e^{-\sigma(M'AM)}dA.$$

The left-hand side has at $s=1$ the residue

$$r=2\pi^{-\frac{1}{2}n(n+1)}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{2}\right)\cdots\Gamma\left(\frac{n}{2}\right)\zeta(2)\zeta(3)\cdots\zeta(n),$$

while the residue of the right-hand side is found to be $r=(n+1)v_n$. *K. Mahler* (Manchester).

Schoeneberg, Bruno. Über die Diskriminante der elliptischen Funktionen und ihre Quadratwurzel. Math. Z. **65** (1956), 16–24.

Put

$$e^{2\pi ir}\prod_{1}^{\infty}(1-e^{2\pi irn})^{24}=\sum_{1}^{\infty}\tau(n)e^{2\pi irn},$$

$$e^{\pi ir}\prod_{1}^{\infty}(1-e^{2\pi irn})^{12}=\sum_{n\equiv 1(2)}\tau_2(n)e^{\pi irn}.$$

D. H. Lehmer [Duke Math. J. **14** (1947), 429–433; MR **9**, 12] showed that if $\tau(n)$ ever vanishes the smallest value of n is a prime; he also showed that $\tau(n)\neq 0$ for $n<3316799$. In the present paper explicit expressions for $\tau(n)$ and $\tau_2(n)$ are obtained in terms of certain positive definite quadratic forms in 8 and 4 variables and certain associated harmonic functions [cf. Schoeneberg, Math. Ann. **116** (1939), 511–523]. It is proved that

$$64\tau(n)=\sum_{Q_{16}(n)=n}n_1{}^4-80n^2\sigma_7(n),$$

$$\sum_{Q_{16}(n)=n}n_1{}^4=144n^2\sigma_3(n)+144\cdot 240\sum_{1}^{n-1}(n-i)^2\sigma_3(n-i)\sigma_3(i),$$

whence

$$(*)\qquad \tau(n)=\frac{n^2}{4}\{9\sigma_3(n)-5\sigma_7(n)\}+9\cdot 240\sum_{1}^{n-1}(n-1)^2\sigma_3(n-i)\sigma_3(i).$$

This implies

$$(**)\qquad \tau(n)\equiv\frac{n^2}{4}\{9\sigma_3(n)-5\sigma_7(n)\}\pmod{9.240};$$

hence if $\tau(p)=0$, $p\neq 2, 3, 5$, it is necessary that $5p^7-9p^3-4\equiv 0 \pmod{2^6\cdot 3^3\cdot 5}$, which implies $p\equiv -1 \pmod{2^2\cdot 3^3\cdot 5}$. (The formula (*) as it stands seems to be incorrect; the coefficient $9\cdot 240$ in the right member should apparently read $9\cdot 60$. This change implies a corresponding change in (**) and the result obtained from it. Incidentally the correct form of (*) had been proved in a different way by Lahiri [Bull. Calcutta Math. Soc. **39** (1947), 33–52, in particular p. 35, formula (7.2); MR **9**, 225].

As for the function $\tau_2(n)$, it is proved that

$$(***)\ \tau_2(n)=n\{2\sigma_1(n)-\sigma_3(n)+48\sum_{2i<n}(n-2i)\sigma_1(n-2i)\sigma_1'(i),$$

where $\sigma_1'(n)$ denotes the sum of the odd divisors of n. It

follows from (***) that

$$\tau_2(n)\equiv n\{2\sigma_1(n)-\sigma^3(n)\} \pmod{48}.$$

Thus if $\tau_2(p)=0$, $p\neq 2$ or 3, then

$$p^3-2p^2-1\equiv 0 \pmod{48}.$$

It is also proved that $\tau_2(n)\equiv\sigma_5(n) \pmod{2^8}$; hence $\tau_2(p)=0$ implies $p\equiv -1 \pmod 8$. *L. Carlitz.*

Schäffer, Juan J. The equation $1^p+2^p+3^p+\cdots+n^p=m^q$. Acta Math. **95** (1956), 155–189.

The author shows that for fixed p, q the title equation has an infinite number of solutions only in the following cases. (I) $q=1$ or $p=3$, $q=2$; when every n gives a solution. (II) $p=1$, $q=2$ or $p=3$, $q=4$; when all the solutions are given by $n=\frac{1}{2}(x-1)$, $m=\frac{1}{2}y$ and $x+y\sqrt{2}=(1+\sqrt{2})^r$. (III) $p=5$, $q=2$; when all the solutions are given by $n=\frac{1}{2}(x-1)$, $m=\frac{1}{2}y(3y^2+1)$, $x+y\sqrt{6}=(3+\sqrt{6})(5+2\sqrt{6})^r$.

In all other cases the number of solutions is bounded by a constant depending only on p. The author conjectures that the only other solutions have $m=n=1$ apart from $p=q=2$, $n=24$, $m=70$ and verifies this for extensive classes of (p, q).

The main line of attack is as follows. The left hand side of the title equation may be written

$$n(n+1)(2n+1)P_p(n)/k(p) \qquad (p \text{ even}),$$
$$n^2(n+1)^2P_p(n)/k(p) \qquad (p \text{ odd}),$$

where $k(p)$ is an integer and $P_p(x)$ is a polynomial in x with integer coefficients which have no common factor. Further, $P_p(x)$ has no factor x, $x+1$ or $2x+1$. For, say, even p the title equation implies that n, $n+1$, $2n+1$ are all perfect qth powers multiplied by bounded factors, the bound depending only on p. (And similarly for odd p.) The resulting equations may then be shown insoluble by a wide variety of methods based on an extensive knowledge of the literature.

Of independent interest is the fact that $P_p(x)$ is a polynomial in $y=x(x+1)$ which, for odd p, is neither of the shape $\{T(y)\}^2$ nor $(Ay+B)\{T(y)\}^2$, where $T(y)$ is a polynomial, except for $p=3$, 5 respectively.

J. W. S. Cassels (Cambridge, England).

Emel'yanov, G. V. On a system of Diophantine equations. Leningrad. Gos. Univ. Uč. Zap. **137**. Ser. Mat. Nauk **19** (1950), 3–39. (Russian)

Following Kamke [Math. Ann. **83** (1921), 85–112; J. Reine Angew. Math. **152** (1922), 30–22] and Mardžanišvili [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **2** (1936), 263–264; **22** (1939), 467–470; Izv. Akad. Nauk SSSR. Ser. Mat. **1937**, 609–631; **4** (1940), 193–214; MR **2**, 250] the author considers the set of equations

$$(1) \qquad \sum_{s=1}^{S} x_s^k=N_k \quad (1\leqq k\leqq n),$$

where N_j are given positive integers and x_j are positive integers to be determined. He writes

$$(2) \qquad N_k=h_kN_n^{k/n} \quad (1\leqq k<n)$$

and supposes that $S\geqq cn^2\log n, 1<l_k\leqq h_k\leqq i_kS^{1-k/n}(1\leqq k<n)$ for some constants c, l_k, i_k. In the first two parts he uses the circle method together with the estimates for trigonometrical sums due to Vinogradoff. He shows that the number of solutions of (1) is

$$\frac{B(h_1,\cdots,h_{n-1})}{h_1\cdots h_{n-1}}N_n^{(S/n)+(n+1)/2}\times\{\mathfrak{S}(N_1,\cdots,N_n)+O(N_n^{-1/c_1n^2\log n})\}$$

where $c_1<0$ is a constant, $\mathfrak{S}$ is a 'singular series' and B depends only on $h_1, \cdots, h_{n-1}$. If there is an $\varepsilon>0$ such that the equations

$$\sum_{s=1}^{S} \xi_s^k=h_k' \quad (1\leqq k\leqq n)$$

have a real positive solution $\xi_1, \cdots, \xi_S$ for all real h_k' in $h_k-\varepsilon\leqq h_k'\leqq h_k$ (with $h_n=1$), then B is greater than a positive constant depending only on n, S (and, presumably, on ε). He shows, further, that the singular series is greater than a constant, and so a solution exists, provided that a further elaborate 'condition of local solubility' is satisfied. In the third part the author adapts Linnik's elementary treatment [Mat. Sb. N.S. **12(54)** (1943), 225–230; MR **5**, 200] of Waring's problem to show that (1) is soluble in positive integers for all S greater than a constant S_0 depending only on n provided that the N_k satisfy (2), (4) and provided that the congruences

$$\sum_{i=1}^{n} y_ii^k\equiv N_k \quad (1\leqq i, k\leqq n)$$

are soluble in integers y_i to every modulus. (This was proved with transcendental methods and an explicit S_0 by Mardžanišvili in the second cited work, where the congruence condition was shown to be necessary).

The reviewer was unable to follow the paper in detail as the author presupposes an intimate knowledge of all the preceding literature and seldom makes explicit which parameters depend on others, or which may be chosen arbitrarily. *J. W. S. Cassels* (Cambridge, England).

Lehmer, D. H. On the Diophantine equation $x^3+y^3+z^3=1$. J. London Math. Soc. **31** (1956), 275–280.

The author shows how the theory of Pell's equation can be used for constructing infinite sequences of integral solutions (x, y, z) of $x^3+y^3+z^3=1$. Every such sequence is obtained from a set of recurrence formulas of the type $x_{k+1}=ax_k-x_{k-1}+b$, $y_{k+1}=ay_k-y_{k-1}+b$, $z_{k+1}=az_k-z_{k-1}+c$, and for every such sequence $1-z_k$ is a constant integral multiple of x_k+y_k. In particular, starting from the known parametric solution $(9t^4, 3t-9t^4, 1-9t^3)$, he derives an infinite sequence of new parametric solutions, the simplest being

$$(2^43^5t^{10}-3^35t^4,\ -2^43^5t^{10}-2^43^4t^7-3^4t^4+3t,\ 2^43^5t^9+2^93^4t^6-3^2t^3+1).$$

N. G. de Bruijn (Amsterdam).

Gupta, Hansraj. Partitions into distinct primes. Proc. Nat. Inst. Sci. India. Part. A. **21** (1955), 185–187.

In order to check a table for $P(n)$, the number of partitions of n into primes, for $n\leqq 300$, the author has computed the first 301 coefficients in the expansion of $\prod_p(1-x^p)=\sum S(n)x^n$. Incidentally, this led to the computation of $R(n)$, the number of partitions of n into distinct primes, for $n\leqq 300$. Both tables, for $R(n)$ and $S(n)$, are given here.

N. J. Fine (Philadelphia, Pa.).

Meinardus, Günter. Partitionen und Teilerfunktionen. Arch. Math. **7** (1956), 52–54.

The author generalizes to n-dimensional lattices some results of Ostmann [Math. Nachr. **13** (1955), 157–160; MR **17**, 239] which yield recurrence formulas for partition functions of general type, using divisor functions. A prototype is the well-known $np(n)=\sum_{k=1}^{n}\sigma(k)p(n-k)$.

N. J. Fine (Philadelphia, Pa.).

Nicol, C. A.; and Vandiver, H. S. On generating functions for restricted partitions of rational integers. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 37–42; errata, 251.

The authors study the coefficients in the expansion of $\prod_{i=1}^{n}(1-zx^i)^{a_i}$ as a polynomial in x, the a_i being non-negative integers not all zero. First, a general recurrence formula is given, and then special results are obtained for $z=1$. Finally, for $a_i=1$ $(1\leqq i\leqq n)$, some arithmetic properties of the coefficients are derived. The von Sterneck number plays an important part. *N. J. Fine.*

Mycielski, Jan. Quelques identités de la théorie analytique des nombres. Colloq. Math. **4** (1956), 68–70.

The author proves twenty number theoretic identities. I state here a few of them

$$\sum_{n=2}^{\infty}\frac{\gamma(n)-1}{n}=1,\quad \sum_{n=2}^{\infty}\frac{\psi(n)}{n^s}=\sum_{n=2}^{\infty}\frac{n}{n^s-1},$$

where $\gamma(n)=\sum_{d|b(n)}1$, $a(n)^{b(n)}=n$ and $a(n)$ is the smallest integer satisfying $a(n)^k=n$, k integer, $\psi(n)=\sum_{d|b(n)}n^{1/d}$. *P. Erdős* (Haifa).

Oberhettinger, F. Note on the Lerch zeta function. Pacific J. Math. **6** (1956), 117–120.

The author gives a new proof of the functional equation for the Lerch zeta function $\Phi(z, s, v)$ defined by analytic continuation of the series $\sum_{m=0}^{\infty}z^m(v+m)^{-s}$. This is done by a simple and direct application of Poisson's summation formula. The convergence difficulties which arise in applying Poisson's formula to the special cases of the Riemann zeta function $\zeta(s)=\Phi(1, s, 1)$ and the Hurwitz zeta $\zeta(s, v)=\Phi(1, s, v)$ do not arise in this more general situation.

By using an appropriate integral representation for $\Phi(z, s, v)$ the author also shows how to derive Hurwitz's formula

$$\zeta(s, v)=2(2\pi)^{s-1}\Gamma(1-s)\sum_{n=1}^{\infty}n^{s-1}\sin(2\pi nv+\pi s/2)$$

as a limiting case of the functional equation for Φ. *T. M. Apostol* (Pasadena, Calif.).

Vause, R. Z. On the distribution of the Jacobian symbols. J. Elisha Mitchell Sci. Soc. **72** (1956), 15–24.

For m prime to 6 define M_{ijk} as the number of triplets c, $c+1$, $c+2$ in a complete residue system (mod m) such that the Jacobi symbols $(c/m)\equiv i$ (mod 3), $([c+1]/m)\equiv j$ (mod 3), $([c+2]/m)\equiv k$ (mod 3). Thus the range of i, j, k may be taken as 0, ±1. Let $\phi(m)$ denote the Euler function, and

$$\phi_3(m)=m\prod_{p|m}\left(1-\frac{3}{p}\right)$$

the Schemmel generalization. For square-free m the values of M_{ijk} are evaluated in terms of ϕ and ϕ_3 and the similarly defined function M_{ij}. For example, in case m has a prime factor of the form $8h+3$ then $M_{ijk}=\frac{1}{3}\phi_3(m)$ for $i, j, k=\pm1$. The function M_{ij} has been treated by A. Brauer [Math. Z. **58** (1953), 226–231; MR **14**, 1064]. The writer also evaluates M_{ij}^d, the number of elements x in a complete residue system (mod m) such that $(x/m)\equiv i$ (mod 3) and $([x+d]/m)\equiv j$ (mod 3). *I. Niven* (Eugene, Ore.).

Geometry of Numbers, Diophantine Approximation

Schmidt, Wolfgang. Über höhere kritische Determinanten von Sternkörpern. Monatsh. Math. **59** (1955), 274–304.

Let S be a star body which is symmetrical in the origin O. If k is a positive integer, let $\Delta^k(S)$ denote the lower bound of the determinants of the lattices Λ that have less than k pairs $\pm X$ of lattice points in S (we assume the existence of one such lattice). The reviewer conjectured [J. London Math. Soc. **26** (1951), 307–310; MR **14**, 624]

$$k\Delta^k(S)\geqq\Delta^1(S),$$

but only proved the conjecture when k is a prime. The author gives a series of theorems, showing that the conjecture holds when k is of various forms, culminating in a theorem (Satz 5) showing that the conjecture holds for all sufficiently large integers k. He also discusses the more general inequality

$$k\Delta^{kl}(S)\geqq\Delta^l(S),$$

showing that this holds when $l=2$ and k is prime, but that it fails for suitable $S=S_{k,l}$ when $l\geqq3$, $k>1$. However he shows that, if S is bounded, then

$$l\Delta^l(S)\leqq\lim_{k\to\infty}k\Delta^k(S),$$

for each positive integer l, the limit on the right having a finite value. This limit is determined in the case when S is a convex plane domain. The inequality

$$k\Delta^k(S)\leqq\frac{V(S)}{2\zeta(n)}\sum_{\nu=1}^{k}\frac{1}{\nu^n},$$

generalizing the Minkowski-Hlawka theorem, is established.

The reviewer remarks that, using the results of A. E. Western's [ibid. **9** (1934), 276–278] examination of J. W. L. Glaisher's factor tables and D. N. Lehmer's list of prime numbers, it is easy to verify that the author's first four theorems prove the conjecture for all $k\leqq10^7$. *C. A. Rogers* (Birmingham).

See also: Davenport, p. 1235; Watson, p. 1235.

ANALYSIS

★ **Lense, J. Grundbegriffe der klassischen Analysis, gewöhnliche Differentialgleichungen, Funktionentheorie. Handbuch der Physik. Bd. I.** Mathematische Methoden I, pp. 1–89. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. DM 72.00.

This is a lucid and rather complete summary that will serve well the references needs for which it is intended. Contents: I. Real functions of a real variable. II. Differential calculus. III. Real functions of several real variables. IV. Integral calculus. V. Infinite series. VI. Functions of a complex variable. VII. Ordinary differential equations. *F. A. Ficken* (Knoxville, Tenn.).

Avdeev, N. Ya. On rational integration in finite form of some binomial integrals. Rostov. Gos. Ped. Inst. Uč. Zap. **1953**, no. 2, 21–29. (Russian)

Certain integrals of the form $\int F(x, (R(x))^{1/n}dx$, where F and R are rational, are reduced by rational substitutions to integrals of rational functions. *E. R. Kolchin.*

See also: Trijtzinsky, p. 1247.

Theory of Sets, Functions of Real Variables

★**Erdélyi, A.; and Swanson, C. A. Asymptotic forms of Coulomb wave functions. II.** Tech. Rep. 5. Department of Mathematics, California Institute of Technology, Pasadena, 1955. 24 pp.

In a preceding report [Erdélyi, Kennedy, and McGregor, same Rep. 4 (1955); MR **17**, 1083] asymptotic representations of solutions of the differential equation

$$(*)\qquad \frac{d^2y}{dx^2}+\nu^2\Big(\frac{x}{1+x}-\frac{L(L+1)}{\nu^2(1+x)^2}\Big)y=0$$

were obtained for $\nu\to\infty$ with $x\geqq-1+\varepsilon$, $\varepsilon>0$. These results were used to obtain asymptotic forms for the Coulomb wave functions $G_L(\eta,\varrho)$ and $F_L(\eta,\varrho)$ under the hypotheses $\varrho>0$, $\eta\to+\infty$ and $\varrho\geqq2\eta$, $\eta\to+\infty$, respectively. In this report the asymptotic form of $F_L(\eta,\varrho)$ is determined when $0<\varrho\leqq2\eta$. Beyond this, simplified forms for $F_L(\varrho,\eta)$ and $G_L(\varrho,\eta)$ are derived when $\varrho\geqq2\eta$ and $\varrho<\varepsilon\eta$, respectively.

To do this the asymptotic behavior of the solution of (*) subdominant at $x=-1$ is determined by comparing (*) with the differential equation satisfied by the function $(X/X')^{\frac{1}{2}}I_p(\nu X)$, where $X=\int_{-1}^{x}[-t/(1+t)]^{\frac{1}{2}}dt$. This solution is then identified with $F_L(\eta,\varrho)$. Connection formulas relating this solution with the solutions of (*) studied in the previous report are obtained. These connection formulas lead to a description of $F_L(\eta,\varrho)$ when $0<\varrho\leqq2\eta$ and $G_L(\eta,\varrho)$ when $\varrho<\varepsilon\eta$. $F_L(\eta,\varrho)$ and $G_L(\eta,\varrho)$ are tabulated for $\varrho=2\eta$ and $L=0$ for values of η between 2 and 200. Some values of these functions are also given for $L=0$, $\eta=3,5$, $\varrho=1(1)10$.

Attention should be called to the fact that A. Ziebur [Thesis, Univ. of Wisconsin, 1950] has derived asymptotic series for solutions of a class of differential equations containing (*). He applied his results to obtain the asymptotic form of the Whittaker function $M_{k,m}(z)$ for all bounded z and m, $\Re(m)\geqq0$, and $|k|>N$, k, m, and z being complex. The function $F_L(\eta,\varrho)$ is a specialization of $M_{k,m}(z)$. Ziebur's result, up to terms of order $k^{-3/2}$, is

$$M_{k,m}(z)=H(z,k,m)+\frac{z^{2m+1/2}O(1)}{k^{m+5/2}},$$

when $|(4kz)^{1/2}|\leqq N$, and

$$M_{k,m}(z)=H(z,k,m)+\frac{(4kz)^{1/4}}{k^{m+5/2}}\left[O(1)e^{4\sqrt{(4kz)}}+O(1)e^{-4\sqrt{(4kz)}}\right],$$

when $|(4kz)^{1/2}|\geqq N$, wherein

$$H(z,k,m)=\frac{\Gamma(2m+1)}{k^{m+1/2}}D^{-1/2}$$

$$\times\Big\{A(kz)^{1/2}J_{2m}(\xi)+\frac{Bz}{\xi}[(1-2m)J_{2m}(\xi)+\xi J_{2m-1}(\xi)]\Big\},$$

$$D=A^2+A(Bz)'k^{-1}-\Big[A'-B\Big(1+\frac{\frac{1}{4}-m^2}{kz}\Big)\Big]Bzk^{-1},$$

$$B=-\frac{z}{12}-\Big(\frac{\frac{1}{4}-m^2}{6}-\frac{z^2}{120}+\frac{z^4}{10{,}368}\Big)k^{-1},$$

$$A=1+\Big(\frac{z}{12}-\frac{z^3}{288}\Big)k^{-1},\quad \xi=2(kz)^{1/2}.$$

N. D. Kazarinoff (Ann Arbor, Mich.).

Kurepa, Đuro. Quelques aspects de l'importance de la théorie des ensembles. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 255–257. (Serbo-Croatian summary)

The author emphasizes the importance of the theory of sets in modern mathematics and underscores the part played by Yugoslav mathematicians in its development, especially in the areas of abstract distance, generalized matrices and the continuum. Emphasis is laid on the extent to which sets can play the role of numbers.

Author's summary.

Kapuano, Isaac. Deux propositions nouvelles équivalentes à l'hypothèse du continu. C. R. Acad. Sci. Paris **242** (1956), 2614–2617.

L'hypothèse du continu est équivalente à chacune des propositions suivantes: A. Il existe un ensemble linéaire analytique ou un complémentaire analytique réunion d'au moins $\aleph_c$ (puissance du continu) ensembles-G_δ deux à deux disjoints effectivement de classe 2. B. Il existe un ensemble linéaire borélien qui n'est pas un F_σ et qui est réunion de $\aleph_1$ ensembles fermés. (Résumé de l'auteur.)

L. Gillman (Lafayette, Ind.).

Froda, Alexandru. Suites "normales" transfinies. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 861–869. (Romanian. Russian and French summaries)

A few obvious results concerning matrices with α rows and β columns (where α and β are transfinite ordinal numbers), whose elements belong to the set $\{0, 1\}$.

F. Bagemihl (Notre Dame, Ind.).

Popadić, Milan S. A new formulation of the principle of induction. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **8** (1955), 29–33. (Serbo-Croatian. English summary)

Let M, S, S_1, N be sets; let φ be a binary relation. This wording of the induction principle is proved (Proposition 2.1). We have $M\subseteq N$, provided that 1) $S_{M\cap N}\cap S_1\neq\Lambda$, $\{\Lambda\}$, 2) there exists a relation $\psi\in P\varphi$ such that

$$D\psi=S_M\setminus\{\Lambda, M\},\ W_{S_{M\cap N}}\setminus\{\Lambda, M\}\psi\subseteq P(M\cap N).$$

Here $S_A=S\cap PA$ (PA=the class of all subsets of A). A binary relation in V is any set in V^2; $D\varphi$ (respectively $W\varphi$) is the class of first (second) components of all the $x\in\varphi$. For any class A, $D_A\varphi(W_A\varphi)$ is the class of all left (right) components of elements of A whose right (left) components are in A. In the previous wording, S, S_1 and φ are called: inductive system, basis of induction and inductor respectively. The ordered quadruple (M, S, S_1, φ) is "potential" provided $\sim W_{S_E}\setminus\{v\}\psi\subseteq PE$ for each $\psi\in P\varphi$ such that $D_\psi=S_M\setminus\{\Lambda, M\}$, $E\subset M\cup\bigcap X\,(X\in W\psi)$, $S_E\cap S\neq\Lambda$, $\{\Lambda\}$. In order that Proposition 2.1 hold, it is necessary and sufficient that (M, S, S_1, φ) be potential and that the union of sets $W\psi$ contains M for every $\psi\in P\varphi$ with $D\psi=S_M\setminus\{\Lambda, M\}$.

Đ. Kurepa (Zagreb).

Sedmak, Viktor. Sur un système des équations ensembliste. Bull. Soc. Math. Phys. Serbie **7** (1955), 217–218 (1956). (Serbo-Croatian. French summary)

Le symbole d parcourant l'ensemble D des entiers, l'A. prouve que le système ensembliste

$$(X_d\cap A_{d+1}')\cup(X_{d+1}'\cap A_d)=A_d$$

équivaut au système

$$A_d\cap X_{d+1}\subseteq X_d\subseteq(A_d\cup A_{d+1})\setminus[A_{d-1}\cap(A_d\cup A_{d-2})]$$

[cf. S. Četković, same Bull. **4** (1952), nos. 1–2, 51–59; MR **14**, 238; Č. Stanojević, ibid. **4** (1952), no. 3–4, 39–41; MR **14**, 854]. *Đ. Kurepa* (Zagreb).

Sierpiński, W. Sur l'équation $\xi^2=\eta^3+1$ pour les nombres ordinaux transfinis. Fund. Math. **43** (1956), 1–2.

No transfinite ordinal numbers ξ, η satisfy the equation $\xi^2=\eta^3+1$. *F. Bagemihl* (Notre Dame, Ind.).

Sierpiński, W. Sur une propriété des nombres ordinaux. Fund. Math. **43** (1956), 139–140.

The necessary and sufficient condition [cf. Bachmann, Transfinite Zahlen, Springer, Berlin, 1955, p. 100; MR **17**, 134] for the validity of the relation $\alpha\beta=\beta\alpha$ for ordinal numbers α, β is derived without the use of the normal form. *F. Bagemihl* (Notre Dame, Ind.).

Jacobsthal, Ernst. Über den Borelschen Überdeckungssatz. Arch. Math. **7** (1956), 197–200.

The author makes the following definition: If M is a subset of a metric space, $M\subset N=\mathbf{U}\Omega$, $\{\Omega\}$ a collection of open sets, and there exists $d>0$ such that for every $x\in M$ there exists Ω such that $x\in\Omega$ and the distance of x from the boundary of Ω is not less than d, then N is a d-covering of M. Further, it is shown that in p-dimensional Euclidean space every d-covering of a bounded set M contains k elements covering M with $k\leqq n^p$, where $n=[D/d]+1$ (D is the diameter of the smallest p-dimensional interval containing M). If M is bounded and closed, then every open covering of M is a d-covering, whence the Borel covering theorem follows. *C. Masaitis.*

Landis, E. M. On the set of points of existence of an infinite derivative. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 202–204. (Russian)

Let E be an $F_{\sigma\delta}$ of measure zero in [0, 1]. Then there exists a function $F(x)$, defined and continuous on [0, 1], such that $F'(x)=+\infty$ if $x\in E$ and the lower derivate of $F(x)$ is less than $+\infty$ if $x\notin E$. *F. Bagemihl.*

Marcus, Solomon. Sur quelques notions de monotonie concernant les fonctions réelles de deux variables réelles. C. R. Acad. Sci. Paris **242** (1956), 2207–2209.

Marcus, Solomon. Sur la structure des ensembles de niveau des fonctions de deux variables. C. R. Acad. Sci. Paris **242** (1956), 2273–2275.

Following on the work of Kronrod [Uspehi Mat. Nauk (N.S.) **5** (1950), no. 1(35), 24–134; MR **11**, 648], the author gives, in connection with the level sets of a real function defined in the unit square, two lists of theorems which concern (a) some seven notions of monotony, and (b) four modifications of the conditions T_1 and T_2 of Banach.

L. C. Young (Madison, Wis.).

Kudryavcev, L. D. On extension of functions and imbedding of classes of functions. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 501–504. (Russian)

This note announces results which are only partly intelligible to the reviewer in the absence of some of the definitions. Thus, the author's first theorem concerns the extension, from an m-dimensional K manifold of class k to Eudidean n-space, of a function given on K together with its partial derivatives up to a certain order $<k-(n-m)/p$ where $p\geqq 1$. The original values of these partial derivatives are assumed to belong to classes H, depending on these various indices and on a further quantity M, classes which the author has neglected to define. The extension, whose existence is asserted, vanishes outside some neighborhood of K and satisfies various conditions which again depend on undefined symbols. *L. C. Young.*

Mel'nik, S. I. Oscillating functions and some of their applications to problems of mathematical physics. Mat. Sb. N.S. **38(80)** (1956), 465–478. (Russian)

The author develops a previous note of his [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 705–708; MR **16**, 79]. New features are (1) "oscillating functions of order m", defined as orthogonal to all polynomials of degree $\leqq m$ over a set of (spatial) regions, (2) use of spaces L^p, L^q in place of L^2, (3) a finite-difference procedure for $y'=f(x, y)$, (4) estimates for the deformation of elastic bodies. In the applications (3), (4), however, only the original case $m=0$ is used. *F. V. Atkinson* (Canberra).

Pauc, Chr. Considérations sur les gradients généralisés de G. Fichera et E. De Giorgi. Ann. Mat. Pura Appl. (4) **40** (1955), 183–192.

Soit f une fonction numérique borélienne et bornée dans R^n. A tout intervalle fermé I de R^n on associe la fonction vectorielle d'intervalles F^*, dont les composantes sont

$$F_i^*(I)=\int_{(F^+)_i} f\,d\sigma_i-\int_{(F^-)_i} f\,d\sigma_i,$$

où $(F^+)_i$ et $(F^-)_i$ sont les faces supérieure et inférieure de I, perpendiculaires à l'axe des x_i, et σ_i la mesure de Lebesgue ($(n-1)$-dimensionnelle) sur ces faces ($i=1, 2, \cdots, n$). L'hypothèse Fichera consiste à supposer que F^* est à variation bornée [Lezioni sulle transformazioni lineari, v. 1, Ist. Mat., Univ. Trieste, 1954; MR **16**, 715]. Lorsqu'elle est réalisée, on notera également F^* la mesure (vectorielle) de Radon associée à la fonction d'intervalles F^*.

Considérons d'autre part la régularisée f_λ de f par la fonction $(\pi\lambda)^{-n/2}\exp(-|x|^2/\lambda)$, avec $\lambda>0$; soit μ_λ la mesure de Radon (vectorielle, absolument continue) de densité grad f_λ; l'hypothèse De Giorgi consiste à supposer que la variation totale des μ_λ est uniformément bornée [Ann. Mat. Pura Appl. (4) **36** (1954), 191–213; MR **15**, 945]; s'il en est ainsi, μ_λ converge faiblement vers une mesure de Radon μ lorsque λ tend vers 0.

L'auteur montre que l'hypothèse Fichera implique l'hypothèse De Giorgi, et alors les mesures F^* et μ sont égales; inversement, si f est continue (restriction indispensable), l'hypothèse De Giorgi implique l'hypothèse Fichera. *J. Deny* (Strasbourg).

César de Freitas, A. Un produit multiplicatif de distributions de Heaviside. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1955–1956), 135–146.

(i) If T is a distribution which at a point a is locally a function f then if f is continuous at a, set $T\delta_a=\delta_a T=f(a)\delta_a$, and if f has a discontinuity of the first kind at a, set $T\delta_a=f(a-)\delta_a$ and set $\delta_a T=f(a+)\delta_a$; (ii) set $\delta_a^{(n)}\delta_b^{(m)}=0$ for all $n\geqq 0$, $m\geqq 0$, and all a, b.

From these definitions and the law $D(ST)=D(S)T+SD(T)$ there follows an extension of the usual notion of product for distributions. If the product ST is defined then TS is necessarily also defined but $ST\neq TS$ in general.

I. Halperin (Kingston, Ont.).

Theory of Measure and Integration

Kuramochi, Zenjiro. An estimation of the measure of linear sets. Proc. Japan Acad. **32** (1956), 105–110.

A set F on the real axis is shown to have linear measure zero provided that a certain condition is fulfilled which involves the conformal moduli of ring domains composed of the differences of regions forming an exhaustion of the complement of F. *P. R. Garabedian.*

Battin, R. H. Note on the "Evaluation of an integral occurring in servomechanism theory". Pacific J. Math. **5** (1955), 481–482.

In close relation to a paper by W. A. Mersman [same J. **2** (1952), 627–632; MR **14**, 455] the author presents some simplifications for evaluating the integral $I=(2\pi i)^{-1}\int_{-\infty i}^{\infty i} g(x)[h(x)h(-x)]^{-1}dx$, where g and h are polynomials in x of order $2n-2$ and n, respectively. In both papers the quantity I is linked to the coefficients of the polynomials. Since Battin's paper cannot be read independently, the reviewer refers the interested reader to a direct comparison of both papers. *H. F. Bückner.*

Leach, Ernest B. On a converse of the Hölder inequality. Proc. Amer. Math. Soc. **7** (1956), 607–608.

Let X be a measure space with measure μ, and consider the associated spaces L^p and L^q, where $1<p<\infty$, $1/p+1/q=1$. The converse of Hölder's inequality states that if f is measurable and if $\int|fg|d\mu<\infty$ for every $g\in L^q$, then $f\in L^p$. It is proved that this converse is true if and only if every measurable subset E of X, of measure $+\infty$, contains a measurable subset F such that $0<\mu(F)<+\infty$. *W. Rudin* (Rochester, N.Y.).

Cohen, L. W. A non-archimedian measure in the space of real sequences. Pacific J. Math. **6** (1956), 9–24.

In the set S of all real sequences $X=(x_n)$ let addition be defined coordinatewise and write $X>(0, 0, \cdots)=0$ if there is m such that both $x_m>0$ and $x_n=0$ for all $n<m$. Let S be topologized by considering the open intervals $\{Z|X<Z<Y\}$ as a basis for the open sets. The author defines a function μ on a ring of subsets of S generated by a basis B of the topology in S so that μ has the following properties: Its values are nonnegative elements of S; it is finitely additive, invariant under translations by elements of S and nonzero on B. *H. M. Schaerf.*

See also: Ohtsuka, p. 1196.

Functions of a Complex Variable, Generalizations

Gaier, Dieter. Über die konforme Abbildung veränderlicher Gebiete. Math. Z. **64** (1956), 385–424.

G_n étant une suite de domaines simplement connexes plans, convergeant vers un domaine G_0, on désigne par $f_n(z)$ [resp. $f_0(z)$] une représentation conforme normalisée de $|z|<1$ sur G_n (resp. G_0). L'A. apporte ici quelques précisions concernant la convergence de la suite f_n vers f_0, dans le cas où les domains considérés ne possèdent que des points frontières accessibles. Divers résultats sont d'abord rappelés; la convergence des domaines G_n est ensuite définie à l'aide de la distance de Fréchet des frontières; le cas des domaines limités par une courbe de Jordan est étudié de façon plus détaillée, la convergence étant assurée moyennant une condition de régularité uniforme vérifiée par les frontières; l'A. étudie ensuite la convergence suivant d'autres normes que celle de la convergence uniforme; puis il établit une condition nécessaire et suffisante pour la convergence continue des fonctions inverses des f_n, et aussi pour leur convergence continue en un sens généralisé. Il termine par quelques remarques relatives au cas où la suite G_n est monotone. *J. Lelong* (Lille).

Kennedy, P. B. Conformal mapping of bounded domains. J. London Math. Soc. **31** (1956), 332–336.

Let $f(z)$ be regular univalent and bounded in $|z|<1$ and let $l(r)$ be the length of the image curve of the segment $(0, r)$ of the real axis. Keogh [same J. **29** (1954), 379–382; MR **15**, 862] has proved that $l(r)=o((-\log(1-r))^{\frac{1}{2}})$ and that the exponent $\frac{1}{2}$ can not be decreased. The author sharpens this by proving that if $\mu(r)$ is any positive function decreasing steadily to 0 as $r\to1$, then there is an $f(z)$ in the class under consideration for which $l(r)>\mu(r)(-\log(1-r))^{\frac{1}{2}}$ for a sequence r_n with $r_n\to1$. *A. W. Goodman* (Princeton, N.J.).

Ahiezer, N. I.; and Kreĭn, M. G. On a generalization of the lemmas of Schwarz and Loewner. Har'kov. Gos. Univ. Uč. Zap. **40**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **23** (1952), 95–101 (1954). (Russian)

Let $w=F(z)$ ($z=x+iy$, $w=u+iv$) be any analytic function mapping the half-plane $y>0$ into (a subset of) $v>0$ and taking real values on the part $(-\infty, a)$ of the boundary (class $S(a, \infty)$). Further let $F(i)=i$. The authors describe a circular segment which must contain the values $F(\zeta)$, where ζ is a point of the upper half-plane. Schwarz's lemma and Loewner's lemma [Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 51] are derived as special cases. This result is the special case $n=1$ of the following theorem: a necessary and sufficient condition for the existence of an element F of $S(a, \infty)$ such that $F(z_\alpha)=w_\alpha$, $\alpha=1, 2, \cdots, n$; $F(z_{n+1})=F(i)=iT=w_{n+1}$ ($T>0$) is that the forms

$$\sum_{\alpha,\beta=1}^{n+1}\frac{w_\alpha-\overline{w}_\beta}{z_\alpha-\bar{z}_\beta}\xi_\alpha\bar{\xi}_\beta,\quad \sum_{\alpha,\beta=1}^{n}\frac{\Omega_\alpha-\overline{\Omega}_\beta}{z_\alpha-\bar{z}_\beta}\xi_\alpha\bar{\xi}_\beta$$

are nonnegative, where $\Omega_\alpha=(w_\alpha-Tz_\alpha)(z_\alpha-a)/(1+z_\alpha^2)$. As a consequence the authors deduce that $F(z)$ belongs to $S(a, \infty)$ if and only if both $F(z)$ and

$$\{F(z)-\Re F(i)\}\frac{z-a}{1+z^2}+\frac{\Im F(i)}{1+z^2}(1+a^2)$$

map $y>0$ into $v>0$. *R. P. Boas, Jr.* (Evanston, Ill.).

Ohtsuka, Makoto. Théorèmes étoilés de Gross et leurs applications. Ann. Inst. Fourier, Grenoble **5** (1953–1954), 1–28 (1955).

Ohtsuka, Makoto. Sur un théorème étoilé de Gross. Nagoya Math. J. **9** (1955), 191–207.

Ohtsuka, Makoto. Sur les ensembles d'accumulation relatifs à des transformations plus générales que les transformations quasi conformes. Ann. Inst. Fourier, Grenoble **5** (1953–1954), 29–37 (1955).

Let f be a mapping of class C' of a domain in a Riemann surface R_1 onto a domain of Riemann surface R_2; f is assumed to have non-vanishing Jacobian in terms of local coordinates. The map is said to have parabolic character relative to a subset of the ideal boundary if certain conditions hold, which generalize those for conformal mapping. Various theorems of the type of the Gross star theorem are proved for such a mapping. With

the tools thus developed, the author establishes several results on analytic functions; in particular, he proves a stronger form of a theorem of Beurling and Tsuji on the set of directions on which a function analytic for $|z|<1$ can have a given radial limit.

In the second paper, a new definition of parabolic character is given, in terms of extremal length, and somewhat stronger results are obtained, along with several theorems on extremal length and on type of a Riemann surface.

In the third paper the generalized Gross star theorem is applied to establish properties of interior and boundary cluster sets of very general mappings; in particular, results of T. Yosida are generalized [Proc. Japan Acad. **27** (1951), 268–274; MR **14**, 365]. *W. Kaplan.*

Ahiezer, N. I.; and Marčenko, V. A. On some questions of approximations of continuous functions on the whole real axis. II. Har'kov. Gos. Univ. Uč. Zap. **29**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **21**(1949), 5–9. (Russian)

[For part I see Zap. Naučno-Issled. Inst. Mat. Meh. Har'kov. Mat. Obšč. (4) **19** (1948), 21–25; MR **12**, 89.] The authors use Poisson's summation formula to construct approximating trigonometric polynomials for entire functions of exponential type σ for which $f(x)|x-i|^{-2r-1}\in L^2(-\infty,\infty)$. The trigonometric polynomials (generalizations of Levitan's polynomials) are expressible in the form

$$S_h^{(r)}(x)=\sum_{k=-\infty}^{\infty} f(x+2k\pi/h)\left\{\frac{2\sin\frac{1}{2}hx}{hx+2k\pi}\right\}^{2r+2},\quad h=\sigma/n.$$

The authors obtain the following estimate: if $|f(x)|\leq A+Bx^{2r}$ then

$$|f(x)-S_h^{(r)}(x)|\leq(A+Bx^{2r})(r+2)\left\{1-\left(\frac{2\sin\frac{1}{2}hx}{hx}\right)^2\right\}.$$

They also show how to express $S_h^{(r)}(x)$ in terms only of $f(0), f'(0), \cdots, f^{(2r+1)}(0)$ and $f(k\pi/\sigma)$. *R. P. Boas, Jr.*

Marčenko, V. A. On some questions of the approximation of continuous functions on the whole real axis. III. Har'kov. Gos. Univ. Uč. Zap. **34**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **22** (1950), 115–125 (1951). (Russian)

Let $\alpha(x)$ be positive, continuous, even, increasing to ∞ for $x\to+\infty$, and greater than 1, and let it satisfy the conditions $\alpha(x+y)\leq\alpha(x)\alpha(y)$ and $\int_{-\infty}^{\infty}(1+x^2)^{-1}\log\alpha(x)dx<\infty$. [The first of these is more restrictive than one might suppose: cf. Cooper, Proc. London Math. Soc. (2) **26** (1927), 415–432.] The author constructs a kernel $K(x)$ such that, for each positive λ, $|K(\lambda x)|<M_\lambda(1+x^2)^{-1}/\alpha(x)$, $\int_{-\infty}^{\infty}K(x)dx=1$ and the Fourier transform of K vanishes for $|x|\geq1$, so that K is a generalized Fejér kernel of as arbitrary rate of decrease as it is possible to achieve. Except for a normalizing factor,

$$K(x)=\prod_{k=1}^{\infty}\left(\frac{\sin\frac{1}{2}a_kx}{\frac{1}{2}a_kx}\right)^2,$$

where $a_k=C_k\beta(k)S^{-1}k^{-2}$, $\beta(x)=\log\alpha(x)$, C_k are chosen so that $C_k>1$, $C_k\to\infty$, and $\sum k^{-2}\beta(k)C_k=S<\infty$. Two applications are given. The first is to a generalization of Levitan's trigonometric polynomials of approximation to an entire function of exponential type: here the entire function F of type σ satisfies $\sup|F(x)/\alpha(x)|<\infty$ and the trigonometric polynomials $S_h(x)\to F(x)$ uniformly on compact sets as $h\to0$; their exponents $\lambda_k(h)$ are such that $\max_k|\lambda_k(h)|\leq\sigma+h$; and $\sup_x|S_h(x)/\alpha(x)|\leq\sup|F(x)/\alpha(x)|$. The second application is to a summation method for the "cardinal" interpolation series based on the values $F(n\pi/\sigma)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Malliavin, Paul. Les théorèmes de Duffin-Schaeffer. C. R. Acad. Sci. Paris **242** (1956), 2204–2207.

The theorem of the title states that a function $f(z)$ which is analytic and of exponential type in $x>0$ is bounded on the positive real axis if it is bounded on a sequence $\{\lambda_n\}$ with $|\lambda_n-n|<L$ and $|\lambda_n-\lambda_m|\geq\delta>0$ for $n\neq m$. [For references see Boas, Entire functions, Academic Press, New York, 1954, p. 191; MR **16**, 914.] A theorem of V. Bernstein states that $\limsup\log|f(x)|=\limsup\log|f(\lambda_n)|$ under the less restrictive hypothesis $|\lambda_n-n|=o(n)$. The author announces a complete solution of the problem of interpolating between these theorems by determining the growth of $f(x)$ from the growth of $f(\lambda_n)$, where $|\lambda_n-n|\leq\varepsilon(n)$ and growth is measured by appropriate functions of $\varepsilon(n)$. More generally, the author deals with meromorphic functions. *R. P. Boas, Jr.*

Ahiezer, N. I.; and Levin, B. Ya. On interpolation for entire transcendental functions of finite degree. Har'kov. Gos. Univ. Uč. Zap. **40**=Zap. Mat. Otd. Fiz.-Mat. Fak i Har'kov. Mat. Obšč. (4) **23** (1952), 5–26 (1954). (Russian)

The authors continue their work [Dokl. Akad. Nauk SSSR (N.S.) **65** (1949), 265–268, 781–784; MR **10**, 693; **12**, 1001] in which they used interpolation formulas to study the growth of entire functions of exponential type less than π that are bounded or grow like a power on a sequence $\{\lambda_k\}$ such that $|k-\lambda_k|\leq L$ and $|\lambda_k-\lambda_m|>\delta>0$ $(m\neq n)$. Now they replace the second condition simply by $\lambda_n\neq\lambda_m$ $(m\neq n)$. In this case the λ_n fall into groups of at most r points such that a group contains, with λ_n, all λ_j such that $|\Re(\lambda_j-\lambda_n)|\leq\varrho<1-2L/r$, where r is the smallest integer not less than $2L$. If $\mu_0, \cdots, \mu_s$ are the points in a given group, the hypothesis $|f(\lambda_n)|\leq M$ is augmented by the hypothesis that the divided differences of orders up to s of the values $f(\mu_0), \cdots, f(\mu_s)$ also do not exceed M in absolute value. Then $f(z)$ is represented by an appropriate interpolation formula from which the authors deduce that $f(z)$ is bounded on the real axis. If boundedness is replaced by growth like a power, another interpolation formula shows that the conclusion is growth like the same power, under the conditions that, for some n,

$$\int_{-\infty}^{\infty}e^{-2n|y|}(1+y^{2n})^{-1}|f(iy)|^2dy<\infty$$

and

$$\int_1^{\infty}u^{-2}\{\log\int_u^{\infty}e^{-nv}y^{-n-2}|f(-iy)|+\log\int_u^{\infty}e^{-nv}y^{-n-2}|f(iy)|\}dy=-\infty.$$

Finally the authors consider interpolation formulas and growth theorems when the condition of distinct λ_n is dropped. Let $M_0\supset\cdots\supset M_p$ be a nested collection of relatively dense sets with inclusion intervals $l_0\leq\cdots\leq l_p$ (terminology as in the theory of almost periodic functions). Let $f(z)$ be an entire function of exponential type such that $|f(\lambda)|\leq M, \cdots, |f^{(l)}(\lambda)|\leq M$ at each point λ of M_j. If the indicator h of f satisfies $h(\pi/2)+h(-\pi/2)<2\pi\sum_{j=0}^{p}1/l_j$ then $f(z)$ is bounded on the real axis. *R. P. Boas, Jr.*

Shah, S. M.; and Singh, S. K. On the derivative of a meromorphic function with maximum defect. Math. Z. **65** (1956), 171–174.

The authors show that if $f(z)$ is a nonconstant meromorphic function of finite order, the sum of whose defects is 2, then (with the usual notation of the Nevanlinna theory) $T(r, f') \sim 2T(r, f)$, $\mu(\infty)=0$, $\delta(f', 0)=1$, $\delta(f', \infty)=0$, and $2\sum \delta(f', b_i) \leq 1$ (where apparently $\{b_i\}$ is any sequence of finite nonzero numbers). This is a corollary of a more general theorem: let

$$\Delta(f', 0)=1-\liminf N(r, 1/f')/T(r, f');$$

then for any sequence of finite a_i, we have

$$\Delta(f', 0) \liminf T(r, f')/T(r, f) \geq \sum \delta(a_i)$$

and

$$\Delta(f', 0)\{2-\mu(\infty)-\delta(\infty)\} \geq \{1+\Delta(f', 0)-\delta(f', 0)\}\sum \delta(a_i).$$

R. P. Boas, Jr. (Evanston, Ill.).

Bernardi, S. D. Note on an inequality of Prawitz. Duke Math. J. **23** (1956), 385–391.

Let S denote the class of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which are regular and univalent in $|z|<1$. The inequality of Prawitz states that if $\alpha \neq 0$ and (*) $(z/f(z))^{\alpha/2}=1+\sum_{n=1}^{\infty} b_n z^n$, then (**) $\sum_{n=1}^{\infty} (2n-\alpha)|b_n|^2/\alpha \leq 1$. The author shows that if (*) is a polynomial of degree N and if equality occurs in (**) then $f(z)=z/\prod_1^N (1-ze^{i\theta_j})^{2/N}$.

On the basis of this theorem the author obtains the 45 functions in the class S for which all the coefficients a_n are integers in an imaginary quadratic field. Here reference should be made to V. Linis [Amer. Math. Monthly **62** (1955), 190–110; MR **16**, 809] and T. Shah [J. Chinese Math. Soc. (N.S.) **1** (1951), 98–107; MR **17**, 141] where the same type of theorem is proved.

A. W. Goodman (Princeton, N.J.).

Kufarev, P. P.; and Semuhina, N. V. On extension of Goluzin's variational method to doubly connected regions. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 505–507. (Russian)

The authors state without proof a very complicated formula for the variation of a function univalent in a ring Q: $r_0<|w|<R_0$. The authors then apply this formula to the problem of maximizing $\Re(f(w_0))$ for fixed w_0 in Q, in the class of functions $f(w)$ regular and univalent in Q with $f(1)=1$ and $f(z)\neq 0$ in Q. They obtain a differential equation for the extremal function, and deduce from this that the extremal function maps Q onto a region obtained from the plane by deleting two curves each of which is composed of a finite number of analytic arcs. *A. W. Goodman.*

Akaza, Tohru. Note on the subregions bounded by the level curves of the Green's function. Sci. Rep. Kanazawa Univ. **3** (1955), 193–197.

L'A. étend d'abord un résultat de J. L. Walsh [Duke Math. J. **20** (1953), 611–615; MR **15**, 310]; puis il étudie le cas où, par une transformation convenable, on se ramène au cas étudié par J. L. Walsh. *J. Lelong* (Lille).

Duffin, R. J. Basic properties of discrete analytic functions. Duke Math. J. **23** (1956), 335–363.

Dans ce mémoire, l'A. étend dans diverses directions un travail du Ref. [Bull. Sci. Math. (2) **68** (1944), 152–180; MR **7**, 149], dont il adopte les définitions: une fonction $f(z)$ définie aux sommets d'un quadrillage plan est dite discrète analytique (préholomorphe au sens du Ref.; d.a. en abréviation) si elle satisfait à

$$Lf(z)=f(z)+if(z+1)-f(z+1+i)-if(z+i)=0$$

en chaque point z du réseau. De plus l'A. utilise de nombreuses références aux travaux sur les fonctions préharmoniques. La notion de fonction duale lui permet d'abord d'exprimer sous forme intégrale la dérivée d'une fonction d.a.; une définition adéquate de l'intégrale du quotient de 2 fonctions d.a. lui permet ensuite de donner une formule analogue à celle de Cauchy, d'abord pour un domaine borné, puis, par passage à la limite, pour le demi-plan supérieur, ce qui introduit une fonction $\theta(z)$. Il étudie ensuite l'équation fonctionnelle $Lf=w$, et la résoud par une formule intégrale; l'étude du cas correspondant à la fonction θ lui permet une étude numérique de cette fonction, qui se rattache d'ailleurs à la fonction de Green telle qu'elle avait été définie par Courant, d'où son comportement asymptotique. La partie suivante du mémoire est consacré aux bipolynomes, c'est à dire aux fonctions qui prennent mêmes valeurs qu'un polynome sur le réseau pair, et mêmes valeurs qu'un autre polynome sur le réseau impair. Après une étude algébrique, l'A. établit un théorème analogue à celui de Liouville. Il étudie ensuite l'analogue de la transformation de Fourier et de l'intégrale de Poisson; enfin il definit un opérateur Z qui présente des analogies avec la multiplication d'une fonction analytique par z, et lui permet de donner un développement de la fonction, analogue à e^{zt}, définie par le Ref. *J. Lelong* (Lille).

Takasu, Tsurusaburo. A complex function theory with generalized Dirac matrices as complex units. I. Yokohama Math. J. **2** (1954), 107–126 (1955).

The author generalizes complex function theory for linear algebras. He gives the details for quarternions and 'hyperbolic" quarternions in a parallel treatment.

J. A. Ward (Holloman, N.M.).

Makar, Ragy H.; and Makar, Bushra H. Further results on algebraic basic sets of polynomials. III. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 295–300.

[For parts I and II, and the terminology, see the same Proc. **57**=Indag. Math. **16** (1954), 306–318, 319–330; MR **15**, 955.] The authors study the effect on the order of an algebraic basic set (1) of adding a constant multiple of $\{z^n\}$ and (2) of taking the inverse set. They also give an inequality for the order of an algebraic monic basic set such that the zeros of p_n all lie in $|z| \leq kn^\lambda$; here a (non-simple) set is called monic if the coefficient of the term of highest degree in each polynomial is unity.

R. P. Boas, Jr. (Evanston, Ill.).

Steiner, Antonio. Die radialen Randwerte einer Klasse analytischer Funktionen in der oberen Halbebene. Rend. Circ. Mat. Palermo (2) **4** (1955), 323–336 (1956).

Let K be the class of all functions f which are analytic in the upper half-plane, which have radial boundary values $f(x)$ almost everywhere on the real axis, with $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$, and which are the Poisson integrals of their boundary values. The author shows that the Fourier transform of $f(x)$ then vanishes on a half-line, and conversely, so that K is identical with the well-known class of functions which are analytic in the upper half-plane and for which $\int_{-\infty}^{\infty} |f(x+iy)|^2 dx$ is bounded independently of y.

Reviewer's comment: The Schwarz inequality, applied to the Poisson integral which represents $f(x+iy)$, shows immediately that $\int_{-\infty}^{\infty}|f(x+iy)|^2dx \leq \int_{-\infty}^{\infty}|f(x)|^2dx$ for all $y>0$. *W. Rudin* (Rochester, N.Y.).

Reade, Maxwell O. The coefficients of close-to-convex functions. Duke Math. J. **23** (1956), 459–462.

Let K_α $(0\leq\alpha\leq 1)$ denote the class of functions $f(z)=\sum_1^\infty a_nz^n$ for which 1) $f'(z)\neq 0$ and

$$2)\qquad \int_{\theta_1}^{\theta_2}\Re\left[1+\frac{zf''(z)}{f'(z)}\right]d\theta > -\pi\alpha,\ z=re^{i\theta},$$

for all $\theta_1<\theta_2$ and for all r, $0\leq r<1$. It is shown that if $f(z)\in K_\alpha$ then $f(z)$ is univalent and

$$|a_n|\leq[1+\alpha(n-1)]|a_1|\ (n=2, 3, \cdots).$$

K_0 consists of convex functions and K_1 consists of close-to-convex functions. Analogous results are obtained for a second class K_α^* of functions $F(z)$ which bear much the same relation to star maps that the close-to-convex functions bear to convex maps. *M. S. Robertson.*

Chamfy, Christiane. Sur les coefficients de certaines fonctions méromorphes dans le cercle-unité. C. R. Acad. Sci. Paris **243** (1956), 225–227.

Let $f(z)$ be meromorphic in $|z|\leq 1$ with p poles in $0<|z|<1$. Let $f(z)$ be real for real z and $|f(z)|\leq 1$ on $|z|=1$. Let $f(0)\neq 0$ and let

$$f(z)=u_0+u_1z+\cdots+u_nz^n+\cdots$$

in the neighborhood of the origin. The author sketches a method whereby necessary and sufficient conditions that $f(z)$ have p poles in $|z|<1$ may be obtained in the form of inequalities involving rational functions of the coefficients u_n. By successive stages she reduces the problem to the situation where the function is both bounded and holomorphic in $|z|\leq 1$ for which the solution is well-known. *M. S. Robertson* (New Brunswick, N.J.).

Fišman, K. M. On a certain representation of meromorphic functions in the unit circle. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 366–369. (Russian)

Let σ be a continuous decreasing function on $[0, 1]$, with $\sigma(1)\geq 0$; let $\alpha_n=\int_0^1\sigma(\varrho)\varrho^{2n}d\varrho$; $E(z)=\sum_{n=0}^\infty z^n/\alpha_n$; $S(\varrho)=\int_\varrho^1\sigma(r)dr$. The author's main result is a representation for any function $F(z)$, meromorphic in $|z|<1$, whose Nevanlinna characteristic function $T(r)$ satisfies

$$\int_0^1\sigma(r)T(r)dr<\infty.$$

This involves a generalized canonical product over the zeros and poles of F, in which each factor $(1-z/w_n)$ is multiplied by a convergence-producing factor

$$\exp\{-U_\sigma(z; w_n)\},$$

where

$$U_\sigma(z; w_n)=\pi^{-1}\iint_{|\zeta|\leq 1}\log|1-\zeta/w_n|E(z\bar{\zeta})\sigma(\varrho)d\varrho d\theta,\qquad \zeta=\varrho e^{i\theta};$$

and in addition the nonvanishing factor

$$\exp\{\pi^{-1}\iint_{|\zeta|\leq 1}\log|F(\zeta)|E(z\bar{\zeta})S(\varrho)d\varrho d\theta.$$

R. P. Boas, Jr. (Evanston, Ill.).

Ramamohana Rao, C. On the zeros of an entire function. J. London Math. Soc. **31** (1956), 268–275.

Let $f(x)$ be defined for $x\geq 1$, nonnegative, and increasing for large x. Write M and m for the lim sup and lim inf of $x^{-\varrho}\int_1^x t^{-1}f(t)dt$, and L and l for lim sup and lim inf of $x^{-\varrho}f(x)$ (where $\varrho>0$). These quantities occur in the theory of entire functions, $f(x)$ being then the number of zeros of a given function in $|z|\leq x$. The author proves a number of inequalities connecting M, m, L and l and gives some consequences for entire functions. Some of the results are: $l\leq\varrho m\leq\varrho M\leq L$; the finiteness of L is equivalent to the finiteness of M; if M is finite, $m=0$ is equivalent to $l=0$; if $m<\infty$, then L and M are both finite and positive, or both 0, or both ∞; if L is finite, $L-l\leq e\{\varrho M(\varrho M-l)\}^{\frac{1}{2}}$. More detailed results are given in terms of functions ϕ, ψ, K defined as follows: $\phi(t)$ is the solution of $e^{\phi(t)}(1-\phi(t))=t$; $\psi(t)$ is the solution of $e^{-\psi(t)}(1+\psi(t))=t$; $K(t)=e^{\phi(t)}+T$. Then if $m<\infty$,

$$\varrho m/[1+\psi(m/M)]\leq\varrho m/[1+\log(\varrho M/l)]\leq l\leq L\leq\varrho Me^{1-(\varrho m/L)}\leq\varrho Me^{\phi(m/M)},$$

and $L+l\leq\varrho MK(m/M)$. If $f(x)$ is the counting function of the zeros of an entire function of finite order ϱ and finite positive type τ, these results imply inequalities involving ϱ and τ, considerably extending a rather fragmentary group of known results [for references see Boas, Entire functions, Academic Press, New York, 1954, p. 16; MR **16**, 914]. *R. P. Boas, Jr.* (Evanston, Ill.).

Reich, Edgar. Schlicht functions with real coefficients. Duke Math. J. **23** (1956), 421–427.

Let S_T denote the class of functions

$$f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots,$$

regular and schlicht in $|z|<1$ and having real coefficients a_k. Let $S(\alpha)$ be the class of functions

$$h(z)=a_1z+a_2z^2+\cdots+a_nz^n+\cdots,$$

regular and schlicht in $|z|<1$, for which all a_k are real, $0<a_1<\alpha$, $|h(z)|<1$.

Several of the known properties of the class S_T have been found through an investigation of the larger class of typically-real functions. In particular it is known that

$$m(z)=\inf_{f\in S_T}|f(z)|=\left|\frac{z}{(1+z)^2}\right| \text{ if } \Re(z+\tfrac{1}{2})\geq 2.$$

The situation is more difficult in the case $\Re(z+1/z)<2$. The author, making use of the differential equation for the slit mappings $h(z)$, dense in $S(1)$, shows that 1) $\lim_{r\to 1}m(re^{i\theta})=\frac{1}{4}$ uniformly with respect to θ, and 2) $m(z)$ assumes a maximum at an interior point of $|z|\leq 1$. A representation theorem for $S(\alpha)$ is also obtained. If $h(z)\in S(\alpha)$, z_0 $(|z_0|<1)$ is fixed, then there exist real numbers β, γ, $0<\beta<h'(0)$, $|\gamma|\leq 1-\beta$ such that

$$(1)\qquad \frac{h(z_0)}{[1+h(z_0)]^2}=\frac{\beta z_0}{(1+2\gamma z_0+z_0^2)}.$$

Conversely, if $0<\beta\leq\alpha$, $|\gamma|\leq 1-\beta$, then there exists an $h\in S(\alpha)$ such that $h(z_0)$ satisfies equation (1). *M. S. Robertson* (New Brunswick, N.J.).

See also: Takasu, p. 1180; Babuška et al, p. 1253.

Analytic Functions of Several Variables

Bagemihl, F. A note on power series and area. Michigan Math. J. 3 (1955–56), 133–135.

Verfasser ergänzt Aussagen früherer Autoren [z.B. Ryll-Nardzewski und Steinhaus, Studia Math. 12 (1951), 159–165; MR 13, 732], die die Kondensation von Singularitäten analytischer Funktionen mittels des Kategorieprinzips behandeln. Er betrachtet einen Banachraum X, dessen Elemente x analytischen Funktionen entsprechen. Genauer soll es eine komplexwertige Funktion $f(x, z)$ geben, die bei festem $x \in X$ in $|z|<1$ holomorph ist und bei festem z aus $|z|<1$ eine stetige Linearform in X darstellt. Der Punkt $e^{i\theta}$ auf $|z|=1$ heiße ein starker Punkt von x, wenn jede Umgebung (bezüglich $|z|<1$) von $e^{i\theta}$ durch $f(x, z)$ auf einen Bereich unendlichen Flächeninhalts (gemessen auf der Riemannfläche) abgebildet wird; andernfalls sprechen wir von einem schwachen Punkt. Dann gibt es eine F_σ-Menge Q von erster Kategorie in X und eine offene Teilmenge G von $|z|=1$, so daß jedes $e^{i\theta} \in G$ für alle $x \in X$ ein schwacher Punkt ist, und jedes $e^{i\theta} \in \{|z|=1\}-G$ für alle $x \in X-Q$ ein starker Punkt ist. Für viele Räume X erkennt man nun leicht, daß G leer ist. — Referent bemerkt, daß sich diese Untersuchungen auch bei F-Räumen („FK-Räumen") und allgemeineren Räumen durchführen lassen [vgl. etwa Zeller, Math. Z. 58 (1953), 288–305, 414–435; MR 14, 1092; 15, 134, 1139]. Verfasser hat folgender Druckfehler bemerkt: On page 134, line 17, "nowhere dense" should be replaced by "of first category". *K. Zeller* (Tübingen).

Ozaki, Shigeo; Ono, Isao; and Umezawa, Toshio. On a general second order derivative. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 5 (1956), 111–114.

The authors obtain a sufficient condition for a pseudo-conformal transformation $w_j=w_j(z_1, z_2)$ ($j=1$, 2, and $\partial(w_1, w_2)/\partial(z_1, z_2) \neq 0$) to be univalent in a hypersphere. Their result is a generalization of the following theorem on one complex variable: Let $f(z)=z+a_2z^2+\cdots$ be regular for $|z|<1$ and suppose that $|f''(z)|<1$ for $|z|<1$, then $f(z)$ is univalent in $|z|<1$. For the case of 2 variables they use the matrix notation and their result is: Let $W=W(Z)$ be analytic for $\|Z\|<1$ and let $dW(u)/dZ=E$ where E denotes the 2-ordered unit matrix. Suppose that $\|d^2W/dZ^2\|<1$ for $\|Z\|<1$, then $W(Z)$ is univalent in $\|Z\|<1$. They mention that the generalization of this result to n variables would involve no essentially new ideas and they further point out the newness of their result for the one variable case. The proof of their theorem makes use of another sufficient condition for univalency obtained by S. Ozaki, S. Kashiwagi and T. Tsuboi [Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4 (1952), 230–237; MR 14, 368]. *W. T. Martin* (Cambridge, Mass.).

Michiwaki, Yoshimasa. Several complex variables and Picard's theorem. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 5 (1955), 77–81.

Using matrix notation and a slightly simplified functional equation, the author gives an alternative proof of the following theorem of Fatou and Bieberbach: There exists an entire function

$$W(Z)=\begin{pmatrix} f(Z) \\ g(Z) \end{pmatrix}=Z+\sum_2 A_n[Z^n]$$

with Jacobian identically one which has an exceptional region (open set). Here

$$Z=\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \text{ and } [Z^n]=Z\times Z\times\cdots\times Z,$$

where the sign $\times$ denotes the Kronecker product. *W. T. Martin.*

Magnus, Arne. On polynomial solutions of a differential equation. Math. Scand. 3 (1955), 255–260 (1956).

If an analytic function $f(z)$ is area preserving, then it must satisfy the equation $|f'(z)|=1$; the solutions are trivial, namely $f(z)=az+b$, where a and b are constants and $|a|=1$. The author studies the corresponding problem in two variables z_1 and z_2, that is, he seeks volume preserving pairs of analytic functions $u=u(z_1, z_2)$, $v=v(z_1z_2)$, or, what is the same, solutions of the partial differential equation (1) $u_{z_1}v_{z_2}-u_{z_2}v_{z_1}=1$. In an earlier paper [Proc. Amer. Math. Soc. 5 (1954), 256–266; MR 16, 463], the author investigated this same problem and discussed solutions of (1). As the author observes in the present paper, if (u, v) satisfies (1), so do $(u+F(v), v)$ and $(u, v+F(u))$ where F is arbitrary. Using this idea he constructs chains of solutions starting with the identity mapping $u=z_1$, $v=z_2$. The author conjectures that all polynomial solutions of (1) may be obtained from the identity mapping by means of such chains where the F's involved are polynomials. In the paper cited above the author showed that if m and n denote the degrees of u and v respectively then the conjecture is equivalent to showing that $m|n$ or $n|m$. In the present paper he settles the question in part by showing that (1) has no polynomial solutions with $(m, n)=1$ and $m\geqq 2$, $n\geqq 2$. His theorem is: Let $u=u(z_1, z_2)$ and $v=v(z_1, z_2)$ denote two polynomials of degrees $m\geqq 2$ and $n\geqq 2$ respectively. If the Jacobian $u_{z_1}v_{z_2}-u_{z_2}v_{z_1}=k=\text{const}$ and m and n are relatively prime, then $k=0$ and there exists a polynomial h of first degree in z_1 and z_2 such that u and v are polynomials in h. *W. T. Martin* (Cambridge, Mass.).

Hasimoto, Keizô; Matuura, Syôzô; and Katô, Sadao. The expansion of power-series for the functions of several complex variables. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 5 (1955), 62–76.

Using matrix notations for analytic functions $w_1(z)$, $\cdots$, $w_k(z)$ of k complex variables z_1, $\cdots$, z_k ($k\geqq 2$), the authors continue the work of S. Ozaki and I. Onô [Sci. Rep. Tokyo Bunrika Daigaku. Sect. A, 4 (1953), 262–270; MR 15, 117] on the study of Taylor and Laurent expansions. They define inner and outer radii of associated convergence (for series involving both positive and negative powers) as $2k^2$ numbers $r_j^{(i)}$, $R_j^{(i)}$, $i, j=1, \cdots, k$ such that each function $w_i(z)$ has a Laurent expansion absolutely and continuously convergent for $r_j^{(i)}<|z_j|<R_j^{(i)}$ and diverges for $|z_j|<r_j^{(i)}$ or $|z_j|>R_j^{(i)}$. They give conditions for numbers $r_j^{(i)}$, $R_j^{(i)}$ to be associated radii. They also define other types of convergence radii, namely inner and outer radii of restricted convergence (for Laurent series), radius of restricted convergence and radius of convergence by norm for Taylor series, and they investigate conditions for each of these, as well as relations between the latter two types. *W. T. Martin* (Cambridge, Mass.).

Peschl, Ernst. Automorphismes holomorphes de l'espace à n dimensions complexes. C. R. Acad. Sci. Paris 242 (1956), 1836–1838.

Die Note des Verf. befasst sich mit den analytischen Automorphismen des n-dimensionalen komplexen Zahlen-

raumes C^n und enthält folgende Sätze. 1) Ist T: $x_j' = h_j(x_1, \cdots, x_n)$, $j=1, \cdots, n$, ein analytischer Automorphismus des C^n, der jede Koordinatenebene $x_j=0$, $j=1, \cdots, n$, auf sich abbildet, so ist $D(T)=\det(\partial \lg x_\nu'/\partial \lg x_\mu)=1$. 2) Es seien A_ν, $\nu=1, \cdots, n+1, n+1$ analytische Ebenen der komplexen Dimension $n-1$, die sich im C^n in allgemeiner Lage befinden. Ist dann T ein analytischer Automorphismus des C^n, der $\mathbf{U}_{\nu=1}^{n+1} A_\nu$ auf sich abbildet, so ist T eine komplex-lineare Abbildung.

Der sehr interessante Beweis von Satz 1 ist ausführlich angedeutet. Er verwendet Methoden der Theorie formaler Potenzreihen (Prinzip der formalen Ausfegung etc.). Satz 2) folgt aus Satz 1). *H. Grauert* (Münster).

See also: Kuramochi, p. 1191.

Harmonic Functions, Potential Theory

Rafal'son, Z. H. On the solution of the biharmonic equation. Leningrad. Gos. Univ. Uč. Zap. 144. Ser. Mat. Nauk 23 (1952), 165–191. (Russian)

The author considers the boundary value problem

$$\Delta\Delta u = f(x, y), \tag{1}$$

$$u\Big|_L = \frac{du}{dn}\Big|_L = 0,$$

for a two dimensional region D bounded by a piecewise smooth contour L, and establishes the representation, valid for any solution of (1),

$$u = (2\pi)^{-1} \iint_D q_{\bar{G}} \log r \, d\zeta d\eta, \tag{2}$$

where $q_{\bar{G}}$ is the (unique) projection on the complement of the harmonic functions in the space of square integrable functions in D, of a solution q of $\Delta q = f(x, y)$. It is shown that conversely the formula (2) represents a solution of (1) in a generalized sense. The author also presents some estimates on the behavior of a variational functional associated with (1). *R. Finn* (Pasadena, Calif.).

Bouligand, Georges. Sur quelques problèmes fonctionnels non linéaires. C. R. Acad. Sci. Paris 241 (1955), 1537–1539.

Il s'agit essentiellement de la recherche des fonctions u harmoniques dans le disque-unité, lorsqu'on se donne les valeurs de $|\text{grad } u|$ à la frontière. L'auteur donne une méthode qui fournit une infinité de solutions, et permet de les obtenir toutes lorsque la donnée-frontière est bornée inférieurement par un nombre strictement positif. Il souligne les difficultés du problème analogue dans R^3.

J. Deny (Strasbourg).

Bouligand, Georges. Surfaces minima et opérateurs linéaires associés. C. R. Acad. Sci. Paris 241 (1955), 1676–1678.

Utilisant le paramétrage de Weierstrass, l'auteur applique les résultats de la note précédente pour définir une relation d'équivalence de caractère géométrique dans l'ensemble des surfaces minima. Il souligne un processus linéaire (sorte de produit de composition) dans lequel rentre la détermination des surfaces d'une même classe d'équivalence, et aussi un problème analogue concernant les surfaces de Goursat [Amer. J. Math. 10 (1888), 187–204]. *J. Deny* (Strasbourg).

Bouligand, Georges. Sur un problème variationnel. C. R. Acad. Sci. Paris 242 (1956), 975–976.

A propos du problème résolu dans l'avant dernière note, l'auteur observe que l'ensemble des fonctions u continuement dérivables sur le disque-unité fermé, telles que $|\text{grad } u|$ soit donné sur la frontière, n'est pas compact pour la topologie de la convergence uniforme. *J. Deny.*

Ivanov, V. K. An integral equation of the inverse problem of the logarithmic potential. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 409–411. (Russian)

Let D be a simply connected domain in the complex z-plane such that $0 \in D$, and suppose that D is bounded by a simple closed curve C. Let $V_e(z)$ and $V_i(z)$ be the external and internal potentials of a uniform mass distribution of density σ over D, and let $z(t)$ be a regular function mapping the unit circle $|t|<1$ conformally on D so that $z(0)=0$. The inverse problem referred to in the title is: given $V_e(z)$ and σ, to determine D and the mapping function $z(t)$.

Writing

$$u_e = -\frac{2}{\pi}\frac{\partial V_e}{\partial z}, \quad z^*(t) = \overline{z(t^{-1})} \quad (|t| \gtrless 1),$$

the author shows that $z(t)$ satisfies the integral equation

$$\sigma z^*(t) = -\frac{1}{2\pi i}\int_\Gamma \frac{u_e[z(t)]}{\tau - t} d\tau \quad (|t|>1),$$

where Γ denotes the unit circumference in the t-plane. A converse theorem is also given, and proofs of the following consequences are outlined. (i) If all the singularities of the analytic continuation of $V_e(z)$ lie in a domain QCD, whose inverse image lies in the circle $|t| \leqq r<1$, then $z(t)$ is regular for $|t|<r^{-1}$. (ii) If

$$V_e(z) = a_0 \log(r^{-1}) + \sum_{k=1}^{n} r^{-k}(a_k \cos k\varphi + b_k \sin k\varphi),$$

then $z(t)$ is a polynomial of degree $n+1$. (iii) If C is given parametrically by $x=x(\psi)$, $y=y(\psi)$, where $t=e^{i\psi}$ on Γ, then the functions $x(\psi)$ and $y(\psi)$ give a stationary value to the integral

$$H = \int_0^{2\pi} [x^2 + y^2 + \frac{2}{\pi\sigma} V_e(x, y)] d\psi,$$

under the restrictions that D is bounded and simply connected and $0 \in D$. *F. Smithies.*

Tôki, Yukinari; and Tarumoto, Kôichi. On the pseudo-harmonic functions. Osaka Math. J. 7 (1955), 103–107.

The authors state the theorem: Let $u(p)$ be pseudo-harmonic on an orientable surface F; then one can define local parameters on F with respect to which F is a Riemann surface and $u(p)$ is harmonic. The reviewer finds the proof insufficient; in particular, there appears to be no basis for concluding that the coordinates, as defined by the authors, are conformally related in overlapping neighborhoods. For the special case in which F is homeomorphic to the plane, the theorem has been established by Morse and Jenkins [Fund. Math. 39 (1952), 269–287; MR 15, 210] and by Boothby [Amer. J. Math. 73 (1951), 512–538; MR 13, 266]. *W. Kaplan* (Ann Arbor, Mich.).

Ohtsuka, Makoto. Sur un espace complet de mesures positives dans la théorie du potentiel. Proc. Japan Acad. 32 (1956), 311–313.

In his work on Newtonian potential theory, Henri

Cartan [Bull. Soc. Math. France, **69**, 71–96 (1941) and **73**, 74–106 (1945); Ann. Univ. Grenoble, **22**, 221–280 (1946)] makes use of the fact that the positive measures of finite energy form a complete metric space. But this is generally true only when the basic space forms a topological group and may be false in other cases, such as an arbitrary Riemann surface. The present paper is an attempt to fill this gap for measures in an arbitrary locally compact space Ω. The principal result is that the positive measures of finite energy taken over a fixed compact subspace of Ω form a complete metric space with respect to the norm $(\mu-\nu, \mu+\nu)^{\frac{1}{2}}$, where (μ, ν) is the energy of the pair of measures $[\mu, \nu]$ with respect to a positive, continuous and symmetric potential kernel for which the potential satisfies a continuity condition known to hold for Newtonian potential and for which the energy satisfies the inequality $(\mu-\nu, \mu-\nu)=(\mu,\mu)+(\nu,\nu)-2(\mu, \nu)\geqq 0$, with equality only for $\mu=\nu$. *B. Lepson* (Washington, D.C.).

Arsove, Maynard G. Normal families of subharmonic functions. Proc. Amer. Math. Soc. **7** (1956), 115–126.

Divers critères de normalité, bien connus pour les familles de fonctions harmoniques dans un domaine Ω, sont étendus aux familles de fonctions δ-sousharmoniques [=différences de deux fonctions sousharmoniques; voir Arsove, Trans. Amer. Math. Soc. **75** (1953), 526–551; MR **15**, 622] continues dans Ω, sous l'hypothèse que les mesures associées satisfassent uniformément à une condition de régularité simple. Cas particulier: l'ensemble des fonctions δ-sousharmoniques de $L^p(\Omega)$ ($p\geqq 1$) dont les mesures associées sont également dans $L^p(\Omega)$, constitue un espace de Banach (pour la norme naturelle); applications aux solutions de $\Delta w+\lambda w=0$ (λ=fonction mesurable bornée) et à la construction du noyau reproduisant dans le cas $p=2$. *J. Deny* (Strasbourg).

Huber, Alfred. On functions subharmonic in a half-space. Trans. Amer. Math. Soc. **82** (1956), 147–159.

L'auteur se base sur la transformation de Weinstein [même Trans. **63** (1948), 342–354; Bull. Amer. Math. Soc. **59** (1953), 20–38; MR **9**, 584; **14**, 749] qui fait correspondre les fonctions harmoniques d'un demi-espace R^n et les fonctions qui dans l'espace R^{n+2} diminué de la variété V ($\xi_n=\xi_{n+1}=\xi_{n+2}=0$) sont harmonique et symétriques par rapport à V. Il en déduit une correspondance entre les fonctions sousharmoniques d'un demi-espace R^n admettant aux points de l'hyperplan-frontière une lim sup $\leqq 0$ et les fonctions sousharmoniques de R^{n+2} entier, symétriques par rapport à V. Grâce à cela il transforme des propriétés des fonctions sousharmoniques dans une couronne sphérique ou hors d'une boule de R^n en propriétés des fonctions sousharmoniques dans un demi-domaine analogue du demi-espace R^{n+2} avec lim sup $\leqq 0$ sur la hyperplan. Cela donne des propriétés de moyenne partiellement connues [Ahlfors, Trans. Amer. Math. Soc. **41** (1937), 1–8], et des développements d'une fonction sousharmonique u dans un demi-espace hors d'une boule lorsque à l'infini $u^+/r^{p-\varepsilon}\to 0$ ($p>0$, quelque sort $\varepsilon>0$). Ceux-ci sont déduits de développements d'une fonction sousharmonique hors d'une boule, qui généralisent la représentation de F. Riesz [voir Brelot, Ann. Inst. Fourier, Grenoble **1** (1949), 121–156; MR **12**, 258] lorsque celle ci ne donne plus d'intégrale finie parce que les masses associées sont trop grandes au voisinage de l'infini.

M. Brelot (Paris).

R. Salinas, Baltasar. Complement to a theorem of Ahlfors-Heins on sub-harmonic functions. Rev. Acad. Ci. Zaragoza (2) **9** (1954), no. 2, 119–125. (Spanish)

The author proves the following theorem. Let $u(z)$ ($\not\equiv -\infty$) be subharmonic for $x>0$, and let $u(z)\leqq H(|z|)$, where $H(r)$ is a nonnegative nondecreasing function such that $\int_1^\infty H(r)r^{-2}dr<\infty$. Let u_0 be the Poisson integral of $H(|t|)$ for the right-hand half-plane, and let

$$\sigma=\inf x^{-1}\{u_0(z)-u(z)\}$$

for $x>0$. Then (A) $r^{-1}u(re^{i\theta})\to -\sigma\cos\theta$ as $r\to\infty$ except for a set of θ's of logarithmic capacity zero; (B) this limiting relation holds for each θ if a set of r's of finite logarithmic length is excluded; (C, D) with $\mu_n(\theta)=\int_0^\infty \exp[u(re^{i\theta})]r^n dr$ or $\sup_r\{r^n\exp[u(re^{i\theta})]\}$, we have

$$n\{\mu_n(\theta)\}^{-1/n}\to e\sigma\cos\theta;$$

(E) with $T_\theta(r)=\sup_{n>0} r^n/\mu_n(\theta)$ we have $r^{-1}\log T_\theta(r)\to\sigma\cos\theta$. For $H(r)=0$, (A) and (B) constitute the Ahlfors-Heins theorem [Ann. of Math. (2) **50** (1949), 341–346; MR **10**, 522]. As a corollary the author infers that if $\mu_n(\theta_n)\leqq m_n$ with $|\theta_n|\leqq\frac{1}{2}\pi-\delta$ then either $\liminf n^{-1}m_n^{1/n}=0$ or $\limsup r^{-1}\log T(r)=\infty$ implies $u(z)\equiv-\infty$ (here $T(r)=\sup r^n/m_n$). *R. P. Boas, Jr.* (Evanston, Ill.).

Kurzweil, Jaroslav. On the single-valuedness of the solution of the modified Dirichlet problem. Časopis Pěst. Mat. **78** (1953), 213–214. (Czech)

Let G be a finite open connected region of the complex plane, with boundary consisting of components $\Gamma_0, \cdots, \Gamma_k$. Let $w=u+iv$ be holomorphic in G. Then the boundary conditions $u=c_i$ on Γ_i ($i=0, \cdots, k$) imply $c_0=c_1=\cdots=c_k=c$ and $w\equiv c+id=$const. The author gives a simple proof of this known result, based on the fact that a closed level-curve of u must contain a zero of $w'(z)$.

F. V. Atkinson (Canberra).

Conner, P. E. The Neumann's problem for differential forms on Riemannian manifolds. Mem. Amer. Math. Soc. no. **20** (1956), 56 pp.

The classical Green's and Neumann's problems for a function on a subdomain of Euclidean n-space are the problems of finding a harmonic function with prescribed values on the boundary for the function and its normal derivative, respectively. The author establishes a generalization of these problems to differential forms on an arbitrary orientable Reimannian manifold of class C^∞.

S. S. Cairns (Urbana, Ill.).

Ninomiya, Nobuyuki. Une correction sur mon travail: "Sur l'intégrale d'énergie dans la théorie du potentiel". J. Inst. Polytech. Osaka City Univ. Ser. A. **6** (1955), 79–82.

On considère des potentiels par rapport à un noyau de composition $K(x)$ dans R^n qui est positif, symétrique ($K(x)=K(-x)$), sommable sur tout compact, continu pour tout $x\neq 0$, ayant une limite ($\leqq\infty$) pour $x=0$. On note $U^\mu(x)=\int K(x-t)d\mu(t)$ le potentiel engendré par la mesure $\mu\geqq 0$.

Dans l'article cité dans le titre [même J. **5** (1954), 97–100; MR **16**, 923] l'auteur montrait que si $K(x)$ satisfait au second principe du maximum (principe de Cartan), il est de type positif; on montre ici que le principe ordinaire du maximum (principe de Maria-Frostman) entraîne le même résultat. D'autre part un lemme de l'article cité, dont la démonstration présentait un point obscur, est remplacé par un énoncé simplifié, débarassé de considéra-

tions relatives à la notion de capacité. Ce lemme concerne les propriétés d'un couple de mesures positives, minimisant l'expression $\int U^\mu d\mu \int U^\nu d\nu / (\int U^\mu d\nu)^2$ parmi les couples de mesures μ et $\nu \geqq 0$ portées par deux compacts disjoints donnés; il s'avère utile dans la théorie du potentiel par rapport à un noyau symétrique [voir l'analyse suivante].

J. Deny (Strasbourg).

Ninomiya, Nobuyuki. Sur le théorème du balayage et le théorème d'équilibre. J. Inst. Polytech. Osaka City Univ. Ser. A. **6** (1955), 83–91.

On conserve les notations de l'analyse précédente, et on fait les mêmes hypothèses sur le noyau K; utilisant sa méthode des „paires minimales", l'auteur établit: (1) Pour que K satisfasse au principe du balayage, il faut et il suffit qu'il satisfasse au second principe du maximum. (2) Pour que K satisfasse au principe de l'équilibre, il faut et il suffit qu'il satisfasse au principe du maximum ordinaire. (3) Si K satisfait au principe du balayage (resp. de l'équilibre) le potentiel balayé d'un potentiel donné d'énergie finie (resp. le potentiel d'équilibre) sur un compact quelconque est bien déterminé dans tout l'espace (bien que la mesure correspondante ne soit pas nécessairement unique).

Un résultat analogue à (1), mais avec des hypothèses différentes sur le noyau, avait été établi par H. Cartan et J. Deny [Acta Sci. Math. Szeged **12** (1950), 81–100; MR **12**, 257]; dans des recherches non publiées, G. Choquet et le rapporteur, utilisant des méthodes très différentes, l'ont étendu à tout noyau de composition (pas nécessairement symétrique). Dans le sens: le principe de l'équilibre entraîne celui du maximum, (2) semble nouveau. Pour établir (3), on dégage ce fait intéressant: si le noyau K satisfait au principe du balayage, il est régulier (en ce sens que si la restriction de U^μ au support de la mesure $\mu \geqq 0$ est continue, U^μ est continu dans tout l'espace).

J. Deny (Strasbourg).

Björck, Göran. Distributions of positive mass, which maximize a certain generalized energy integral. Ark. Mat. **3** (1956), 255–269.

L'auteur étudie les mesures $\mu \geqq 0$ de masse totale 1, portées par un compact donné F de R^n $(n>1)$, qui rendent maximum "l'intégrale d'énergie":

$$I(\mu) = \int\int |x-y|^\lambda d\mu(x) d\mu(y),$$

où λ est un nombre >0 donné (dans le cas classique $-n<\lambda\leqq 2-n$, avec égalité exclue si $n=2$, on cherche à rendre $I(\mu)$ minimum). Il existe au moins une μ maximale; une telle μ est portée par la frontière de F (et même par les points extrémaux de l'enveloppe convexe de F pour $\lambda>1$); son potentiel (relativement au noyau $|x^\lambda|$) vaut $I(\mu)$ en tout point du support de μ et est $\leqq I(\mu)$ sur F. Cette mesure est unique pour $0<\lambda<2$, mais non en général pour $\lambda \geqq 2$, cas qui est étudié en détail; pour $\lambda>2$ on a ce résultat curieux: toute mesure maximale est constituée par au plus $n+1$ masses ponctuelles. L'article se termine par quelques remarques concernant d'autres problèmes d'extremum; on y signale de nouvelles divergences avec le cas classique.

J. Deny (Strasbourg).

Bertolini, Fernando. Sul problema di Cauchy per l'equazione di Laplace in tre variabili indipendenti. II. Ann. Mat. Pura Appl. (4) **40** (1955), 121–128.

Les résultats de la partie I de ce travail [mêmes Rend. (8) **16** (1954), 615–624; MR **16**, 923] sont généralisés en supprimant l'hypothèse restrictive que la partie de la frontière qui ne porte pas les données de Cauchy est plane.

J. Deny (Strasbourg).

Shapiro, Victor L. Generalized L_2-Laplacians. Proc. Amer. Math. Soc. **7** (1956), 630–635.

Given $F(x, y)$ in L_2 on the plane, $F(x, y)$ is said to have $f(x, y)$ as a generalized L_2-Laplacian if

$$8t^{-2}\left[\pi^{-1}\int\int_{u^2+v^2<1} F(x+tu, y+tv)\, dudv - F(x, y)\right]$$

tends to $f(x, y)$ in the L_2-norm as $t\to 0$. Theorem: $F(x, y)$ has $f(x, y)$ as a generalized L_2-Laplacian if and only if for every $R>0$ there exists a harmonic function $h_R(x, y)$ such that

$$F(x, y) = \frac{1}{2\pi}\int\int_{u^2+v^2<R^2} \log\left[(x-u)^2+(y-v)^2\right]^{-1} f(u,v)\, dudv + h_R(x, y)$$

almost everywhere in $x^2+y^2<R^2$. An analogous theorem is proved with Euclidean n-space in place of the plane. Also, L_2 can be replaced by L_p if $1\leqq p<2$.

W. Rudin (Rochester, N.Y.).

Albrecht, R. Das Potential in doppelt gekrümmten Kondensatoren. Z. Naturf. **11a** (1956), 156–163.

A series expansion including third order terms is given for the potential produced by types of toroidal, i.e. double-curved, condensers, which are of special technological interest. Edge effects are disregarded in these calculations. Another problem considered is the determination of the shape of the toroidal condenser when the field is prescribed.

J. E. Rosenthal.

See also: Wegner, p. 1253.

Series, Summability

James, R. D. Summable trigonometric series. Pacific J. Math. **6** (1956), 99–110.

The author continues his work on the Fourier character of trigonometric series which at each point are either convergent or summable by a suitable Cesàro method to a function which is everywhere finite. In a previous paper [Canad. J. Math. **2** (1950), 297–306; MR **12**, 94] he gave a definition of integral which makes an everywhere convergent trigonometric series a Fourier series [the first definition of this kind had been given by Denjoy; see, e.g., his Leçons sur le calcul des coefficients d'une série trigonométrique, t. I–IV, Gauthier-Villars, Paris, 1941, 1949; MR **8**, 260; **11**, 99] but the author's approach is different and based on the idea of major and minor functions). In the present paper he considers trigonometric series

$$(*) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

summable (C, k) at each point, where k is a positive integer, and shows that if $(*)$ also satisfies an additional condition (which drops out in the case $k=0$ but not for general k) then (1) the (C, k) sum $f(x)$ of $(*)$ is P^{k+2} integrable [for the definition of P^α integrability see James, Trans. Amer. Math. Soc. **76** (1954), 149–176; MR **15**, 611], (2) the series $(*)$ is the Fourier series of f, in the sense that the a_k and b_k are given by suitable extensions

of Fourier's classical formulas. The additional condition he requires is that that the $(C, k-2)$ means of the series (*) integrated termwise be $o(n)$ at each x. Extensions are given to the case when (*) ceases to be summable (C, k) in a denumerable set. *A. Zygmund* (Chicago, Ill.).

Gaier, Dieter. On the change of index for summable series. Pacific J. Math. **5** (1955), 529–539.

Denote the partial sums of $a_0+a_1+a_2+\cdots$ and $0+a_0+a_1+\cdots$ by s_k and t_k. Let $V(x; s_k)$, $V(x; t_k)$ denote the transforms of $\{s_k\}$, $\{t_k\}$ by means of a linear summability method V with parameter x. Consider the conditions (a) $V(x; s_k)\cong K\cdot x^q$, (b) $V(x; t_k)\cong K\cdot x^q$ for $x\to x_0$, where K is a constant and $q\geqq 0$ is fixed. The author shows that (a) and (b) are equivalent in case V is the Cesàro, Abel, or general Euler transform. For the case in which V is the Borel method he shows that (a) implies (b), and that (b) implies (a) if $\limsup |a_n|^{1/n}<\infty$. He specifies a sense in which the latter condition is best possible. *J. D. Hill.*

Sengupta, H. M. Rearrangements of series. Proc. Amer. Math. Soc. **7** (1956), 347–350.

Let E be the metric space in which a point x is a permutation $(x_1, x_2, x_3, \cdots)$ of the positive integers 1, 2, 3, $\cdots$ and the distance between two points x and y is defined by

$$d(x, y)=\sum_{n=1}^{\infty} 2^{-n}\frac{|x_n-y_n|}{1+|x_n-y_n|}.$$

Let $c(1)+c(2)+\cdots$ be a given real conditionally convergent series. To each rearrangement $c(x_1)+c(x_2)+\cdots$ of the given series there corresponds the point $(x_1, x_2, \cdots)$ of E. Let A denote the proper subset of E which contains those points $(x_1, x_2, \cdots)$ for which the rearrangement $c(x_1)+c(x_2)+\cdots$ of the given series is convergent and, for each x in A, let $f(x)$ denote the real number to which the corresponding rearrangement is convergent. While the set A and the function f depend upon the given series, the author obtains several properties of A and f which are independent of the given series. For example, A is a dense boundary subset of E. *R. P. Agnew* (Ithaca, N.Y.).

Rajagopal, C. T.; and Vijayaraghavan, T. One-sided Tauberian theorems for Borel, Abel and Riemann-second-order transforms. Rend. Circ. Mat. Palermo (2) **4** (1955), 309–322.

Soit $\{s_n\}$ une suite de nombres réels. $\tau(x)=\sum_0^\infty c_n(x)s_n$ une transformation linéaire régulière de $\{s_n\}$ avec $c_n(x)\geqq 0$. De certaines conditions du type tauberien, les auteurs déduisent que $\tau(x)<O(1)$ implique l'ordre de grandeur de s_n à une constante $O(1)$ près. I. $\varphi(u)>0$, $u\geqq 1$, $\varphi(u)\uparrow\infty$; $\Phi(u)=\int_{u_0}^u dt/\varphi(t)\to\infty$, $u\to\infty$. Soit x défini par

$$\Phi(x)-\Phi(M)=\mu\Phi(\lg M),\ \mu>0,$$

et $c_n(x)\geqq 0$, tel que $\Phi(M)\sum_0^M c_n(x)=O(1)$,

$$\sum_{M+1}^{\infty}|\Phi(n)-\Phi(x)|c_n(x)=O(1),\ M\to\infty,$$

alors de $(s_n-s_{n-1})\varphi(n)\geqq -1$ résulte $\tau(x)-s_M\sum_M^\infty c_n(x)\geqq -\mu\Phi(\lg M)+O(1)$. II. Sous une condition plus restrictive du type tauberien portant sur les s_n, de $\tau(x)<O(1)$ résulte $s_n<O(1)$. Applications de ces résultats à la sommabilité du type Abel, Borel et Riemann (du second ordre). [Voir la référence au dessous.] *M. Tomić* (Beograd).

Vijayaraghavan, T.; and Rajagopal, C. T. On two Tauberian theorems for the Borel transform of a sequence. Proc. Indian Acad. Sci. Sect. A. **43** (1956), 163–172.

Dans la note analysée ci-dessus les auteurs ont établi les deux théorèmes suivants relatifs à la sommabilité de Borel. De (1) $B(x)=\sum_0^\infty e^{-x}s_nx_n/n!<O(1)$, $x\to\infty$ et (2) $s_n-s_{n-1}\geqq -\frac{1}{2}W/\sqrt{n}$, $n\geqq 1$, $0\leqq W<\infty$ résulte

$$s_M\leqq\tfrac{1}{2}W\sqrt{\lg M}+O(1),\ M\to\infty.$$

Si (2) est remplacé par (3) $\liminf (s_n-s_{n-1})\sqrt{(n\lg n)}=-W$, $0\leqq W<\infty$, $n\to\infty$, de (1) résulte

$$\liminf\{B(M+2\mu\sqrt{(M\lg M)})-s_M\}\geqq -2\mu W,\ \mu\geqq\tfrac{1}{2},\ M\to\infty.$$

Dans cette note ils montrent seulement, à l'aide des contre-exemples, que les conditions tauberiennes (2) et (3) ne peuvent pas être améliorées. Par exemple, il existe une suite s_n qui satisfait à (1) et (2) et telle que pour une suite convenablement choisie $M_r\uparrow\infty$, $r\to\infty$, on a

$$s_{M_r}=\tfrac{1}{2}W\sqrt{\lg M_r}+O(1),\ r\to\infty.$$

M. Tomić (Beograd).

Peyerimhoff, Alexander. On convergence fields of Nörlund means. Proc. Amer. Math. Soc. **7** (1956), 335–347.

A Nörlund summability method $N(p)$ is defined by the transformation $\sigma_n=P_n^{-1}\sum_{k=0}^n p_{n-k}s_k$, $P_n=\sum_{k=0}^n p_k$, associated with the analytic element $p(z)=\sum p_kz^k$. The convolution $N(q)=N(p)*N(r)$ is the method associated with the element $q(z)$ obtained by formal multiplication of the series of $p(z)$ and $r(z)$. It is shown that for "smooth" $r(z)$ the summability field of $N(q)$ may be computed from that of $N(p)$. More exactly, if $N(p)$ is regular, $r(z)=\sum r_nz^n$ is analytic for $|z|\leqq 1$, $r(0)\neq 0$, $r(1)\neq 0$, and if $r(z)$ has roots α_i $(i=1, \cdots, m)$ in $|z|<1$ with multiplicities γ_i, then s_n is $N(q)$ summable if and only if

$$s_n=t_n+\sum_i \alpha_i^{-n}\sum_{j=\lambda_i}^{\lambda_i+\gamma_i-1} C_{ij}\binom{n+j}{n},$$

where C_{ij} are constants, λ_i is the multiplicity of α_i as root of $p(z)$, and t_n is $N(p)$ summable. A composition theorem holds for absolute summability. Consideration of roots of certain polynomials permits to isolate a finite number of sequences which are added to the summability field in the passage from Cesàro means C_{2k+1} to discontinuous Riesz means K_n^{2k+1} with integral k. *G. G. Lorentz.*

Wilansky, Albert; und Zeller, Karl. Abschnittsbeschränkte Matrixtransformationen; starke Limitierbarkeit. Math. Z. **64** (1956), 258–269.

This is in large part an exppository paper concerning normal (i.e. having the properties $a_{nk}=0$ for $k>n$, $a_{nn}\neq 0$) methods $\lim_n\sum_k a_{nk}x_k$ which have the property of section-boundedness:

$$(*)\qquad \sup_{m,n}|\sum_{k=0}^m a_{nk}x_k|<+\infty$$

for each A-summable sequence x_k. Methods of this type were considered by Jurkat and Peyerimhoff [Math. Z. **55** (1951), 92–108; MR **13**, 934], Peyerimhoff [ibid. **55** (1951), 23–54; MR **13**, 933] and especially Zeller [ibid. **55** (1951), 55–70; MR **13**, 934] who has connected this property with the weak section-convergence. The authors consider: sufficient conditions, and necessary conditions for (*); prove the existence of a base in the summability field of a method satisfying (*); give applications to comparision theorems and convergence factors. Finally, they show that ε_n $(n=0, \cdots)$ are convergence factors of the strong C_1-summability [i.e. $\sum\varepsilon_nx_n$ converges for all sequences x_n with $|x_0-a|+\cdots+|x_n-a|=o(n)$] if and only if $\varepsilon_n\to 0$ and $\sum_{k=0}^\infty|\varepsilon_{n_k}|(n_k-n_{k-1})<+\infty$, where $n_{-1}=-1$ and n_k is the smallest number greater than n_{k-1} such that $|\varepsilon_{n_k}|>|\varepsilon_n|$ for all $n>n_k$. *G. G. Lorentz.*

Kangro, G. On extension of Peyerimhoff's method to double series. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 629–632. (Russian)

Let A, B be summability methods of double sequences generated by four-dimensional matrices, let a sequence x_{mn} be called A' summable if its A-sums converge and are bounded. We call ε_{mn} a summability sequence of class (A', B') if $\sum \varepsilon_{mn}x_{mn}$ is B' summable for each A' summable sequence x_{mn}. Assuming that A is normal and section-bounded [i.e. $a_{mn\mu\nu}=0$ for $\mu>m$ and for $\nu>n$, $a_{mnmn}\neq 0$ and that $\sum_{\mu,\nu=0}^{k,l} a_{mn\mu\nu}x_{\mu\nu}=O(1)$ for each A-summable x_{mn}, cf. the preceding review] the author gives a representation for the ε_{mn} similar to one obtained by Peyerimhoff [Math. Z. **55** (1951), 23–54; MR **13**, 933] for simple sequences. There are further results of this type and applications to special methods. *G. G. Lorentz.*

Zeller, Karl. Über den perfekten Teil von Wirkfeldern. Math. Z. **64** (1956), 123–130.

The summability field of a conservative method A given by a transformation $\sum_k a_{nk}x_k$ is in a natural way a metrizable linear topological space (an "FK-space") $\mathfrak{A}$ which contains the subspace of convergent sequences $\mathfrak{S}_C$. The "perfect part" $\mathfrak{A}_p$ of $\mathfrak{A}$ is defined as the closure of $\mathfrak{S}_C$ in the natural topology. If $\mathfrak{A}_p=\mathfrak{A}$, then A is perfect in the sense of Mazur and Banach. The author proves that for each regular A there is another regular and perfect method BCA with summability field $\mathfrak{A}_p$; he remarks that the general problem of characterizing FK-spaces which are summability fields remains open. There are several instructive examples of cases when the perfect part of a summability field may be effectively described. For instance, the "n-folgeverfahren" and similar methods given by Wilansky and Zeller [Trans. Amer. Math. Soc. **73** (1955), 501–509; MR **16**, 690] have the property $\mathfrak{A}_p=\mathfrak{S}$. *G. G. Lorentz* (Detroit, Mich.).

Endl, Kurt. Über Klassen von Limitierungsverfahren, die die Klasse der Hausdorffschen Verfahren als Spezialfall enthalten. Math. Z. **65** (1956), 113–132.

For each $k=1, 2, 3, \cdots$ there is defined a class $\mathfrak{M}^{(k)}$ of matrices such that $\mathfrak{M}^{(k)}$ reduces to the familiar Hausdorff class $\mathfrak{H}$ when $k=1$. Let k be a fixed positive integer. Let $d_{mn}^{(k)}=1$ when $m\equiv n \pmod k$ and let $d_{mn}^{(k)}=0$ otherwise. Let $\delta^{(k)}$ be the matrix with elements $\delta_{mn}^{(k)}$ defined by

$$\delta_{m,n}^{(k)}=(-1)^{[n/k]}\binom{[m/k]}{[n/k]}d_{mn}^{(k)},$$

where $[q]$ denotes the greatest integer in q. Without the assumption that the diagonal elements constitute a moment sequence, each diagonal matrix μ then determines the matrix $(M^{(k)}, \mu)$ defined by $(M^{(k)}, \mu)=\delta^{(k)}\mu\delta^{(k)}$. For each k, let $\mathfrak{M}^{(k)}$ denote the class of matrices representable in the form $(M^{(k)}, \mu)$, and let $\mathfrak{R}^{(k)}$ denote the subclass of $M^{(k)}$ consisting of the regular matrices in $\mathfrak{M}^{(k)}$. The classes $\mathfrak{M}^{(k)}$ and $\mathfrak{R}^{(k)}$ are studied in considerable detail. Numerous properties of matrices in one class, and relations between matrices in different classes, are obtained. When $k=1$, the matrix $\delta^{(k)}$ reduces to the difference matrix δ (or Δ) of the Hurwitz-Silverman and Hausdorff theory. For each $k=1, 2, 3, \cdots$ the square of $\delta^{(k)}$ is the identity, and $\delta^{(k)}$ has other features which permit the development of a theory of the class $\mathfrak{M}^{(k)}$ which is quite similar to the theory of the class $\mathfrak{H}$. Considerable information is obtained by applying the matrices to power series. When $k=1$, no matrix R in $\mathfrak{R}^{(k)}$ can evaluate the geometric series $\sum z^k$ outside a circle C_R which lies in the halfplane for which $z=x+iy$ and $x\leq 1$. However, when $k\geq 2$, some matrices R in $\mathfrak{R}^{(k)}$ evaluate $\sum z^k$ inside Cassinian ovals which contain points in the halfplane $x>1$. Statements in the introduction and in Theorem 2 that two matrices of the form $\delta\mu\delta$ are consistent (verträglich) should be revised to state that two regular matrices of the form $\delta\mu\delta$ are consistent. *R. P. Agnew.*

Goffman, Casper; and Petersen, G. M. Consistent limitation methods. Proc. Amer. Math. Soc. **7** (1956), 367–369.

Let F denote the family of matrix methods for evaluation of sequences which is defined by the condition that A is in F if (1) A is regular; (2) $a_{mn}\geq 0$; and (3) for each m there is at least one n for which $a_{mn}=\frac{1}{2}$ and for each n there is at least one m for which $a_{mn}=\frac{1}{2}$. It is shown that if a real bounded sequence is evaluable by a method A in F, then it is evaluable to the arithmetic mean of its inferior and superior limits. Moreover it is shown that each real bounded sequence is evaluable by at least one A in F. Thus F is an explicit example of a family of matrix methods which is consistent over the class of bounded sequences and is such that each bounded sequence is evaluable by at least one member of F. While the above family F is consistent over the class of bounded sequences, it is not, as the title of the paper might indicate, a consistent family because some unbounded sequences are evaluable to different values by different members of F. For example if $\theta\geq 0$ and A_θ is the matrix for which $a_{mn}=\frac{1}{2}$ when $n=m$, $m+1$ and $a_{mn}=\theta/m$ when $n=4^m$, then A_θ is in F and the sequence for which $s_m=(-1)^m \log m$ is evaluable A_θ to $\theta \log 4$. *R. P. Agnew.*

Tanaka, Chuji. Note on Dirichlet series. XV. On G. Valiron's method of summation, and Borel's directions. Yokohama Math. J. **2** (1954), 151–164 (1955).

Let $0\leq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty$ and let

$$F(s)=\sum_1^\infty a_n \exp(\lambda_n s),$$

where $s=\sigma+it$. Under the assumption that this series converges uniformly in the whole plane and is of positive order ϱ, i.e. $\limsup_{\sigma\to+\infty} \log^+\log^+ M(\sigma)$ where $M(\sigma)=\sup_{-\infty<t<\infty}|F(\sigma+it)|$, it is shown that there exists a positive increasing function $W(x)$ $(1\leq x<\infty)$ such that $\int^\infty W(x-h)W(x)^{-1}dx<\infty$ for every $h>0$, and such that if

$$\Omega(u)=\int_1^\infty \exp(ux)W(x)^{-1}dx,$$

$$f(s)=\sum_1^\infty a_n\Omega(\lambda_n)\exp(\lambda_n s),$$

then $f(s)$ has abscissa of uniform convergence equal to 0 and $f(s)=\int_1^\infty F(x+s)W(x)^{-1}dx$ for $\operatorname{Re} s<0$. This result is an improvement over earlier treatments of the same subject in that no condition of the form

$$\limsup_{n\to\infty} n/\lambda_n<\infty$$

is imposed. Several applications to the theory of Borel's directions are given. *I. I. Hirschman* (St. Louis, Mo.).

Brauer, George. Evaluation of product sequences by matrix methods. Amer. Math. Monthly **63** (1956), 323–326.

Verfasser betrachtet permanente Matrixverfahren A mit der Eigenschaft P: A limitiert mit zwei beschränkten

Folgen s_n und s_n' stets auch das Produkt $s_n s_n'$ zum „richtigen" Wert (alle Zahlen seien reell). Es genügt, diese Eigenschaft mit den Einschränkungen $A\text{-}\lim s_n=0$ und $s_n=s_n'$ zu fordern. Hat A die Eigenschaft P, so ist $A\text{-}\lim s_n$ bei beschränktem s_n ein Häufungspunkt dieser Folge. Daraus folgt, daß A nicht die Boreleigenschaft hat, also nur „wenige" Folgen aus Nullen und Einsen limitiert [vgl. Hill, Ann. of Math. (2) **46** (1945), 556–562; MR **7**, 153]. Analog zu einem Resultat über die Eigenschaft P für beliebige (auch unbeschränkte) Folgen [siehe etwa Mazur und Orlicz, Studia Math. **14** (1954), 129–160; MR **16**, 814] vermutet Verfasser, daß eine Matrix A ohne leere Spalten die Eigenschaft P nur besitzt, wenn sie konvergenzgleich ist. Dies wird widerlegt durch Verfahren, die außer konvergenten Folgen nur gewisse unbeschränkte Folgen limitieren [siehe etwa Wilansky und Zeller, Trans. Amer. Math. Soc. **78** (1955), 501–509; MR **16**, 690], und durch Verfahren, die der starken Limitierbarkeit äquivalent sind [Zeller, Math. Z. **59** (1953), 271–277; MR **15**, 618]. *K. Zeller* (Tübingen).

Andree, Richard V.; and Petersen, Gorden M. Matrix methods of summation, regular for p-adic valuations. Proc. Amer. Math. Soc. **7** (1956), 250–253.

Die Verfasser betrachten Konvergenz bezüglich einer p-adischen Bewertung $\phi_p=\phi$. Sie zeigen, daß ein dreieckiges Matrixverfahren in diesem Sinne genau dann regulär ist, wenn $\lim_m \phi(a_{mk})=0$, $\lim_m \phi(\sum_t a_{mt}-1)=0$, $\phi(a_{mn})\leqq M$, gilt. $\sum_n \phi(a_{mn})\leqq M$ ist hier — im Gegensatz zu der Zeilennormbedinging bei gewöhnlicher Permanenz — nicht notwendig, weil ϕ eine nichtarchimedische Bewertung ist. *K. Zeller* (Tübingen).

Moser, Leo; and Wyman, Max. Asymptotic expansions. Canad. J. Math. **8** (1956), 225–233.

The authors find an asymptotic expansion for B_n defined by

$$\sum_0^\infty B_n x^n/n!=e^{P(x)},$$

where $P(x)$ is a polynomial in x with positive coefficients. Only the dominant term of the asymptotic expansion had been found earlier by Pólya [Math. Z. **12** (1922), 36–60]. The authors make a skilful use of contour integration to prove their result. *S. Chowla.*

Moser, Leo, and Wyman, Max. An asymptotic formula for the Bell numbers. Trans. Roy. Soc. Canada. Sect. III. (3) **49** (1955), 49–54.

The so-called Bell numbers G_n are defined by

$$\sum_0^\infty G_n z^n/n!=e^{e^z-1}.$$

While Knopp [Theory and application of infinite series, Blackie, London-Glasgow, 1928, p. 563] obtained only the dominant term in the asymptotic expansion of G_n, the authors use contour integration to obtain the complete asymptotic formula for G_n. *S. Chowla.*

★**van der Corput, J. G. Asymptotic expansions. II. Elementary methods.** Department of Mathematics, University of California, Berkeley, Calif., 1955. ii+54 pp. (multilithed).

[For part I (1954) see MR **16**, 352.] In section 1 the theory Pólya and Szegö [Aufgaben und Lehrsätze aus der Analysis, Bd. I, Springer, Berlin, 1925, p. 26] of enveloping series is extended to series of complex terms in the following way: $|\zeta|<\pi$; E_ζ is the closed circle segment bounded by the line segment (0, 1) and by the circle arc with end-points 0 and 1. Which makes at the origin an angle $-\zeta$ with the + real axis. A number s is enveloped by the formal series $\sum a_n$ ($n\geqq 0$) with respect to the circle segment $\sum_\zeta$ if it is possible to find for each integer $n\geqq 0$ an element θ_n of E_ζ such that $s=a_0+a_1+\cdots+a_{n-1}+\theta_n a_n$. Notation: $s\propto\sum_0^\infty a_n(E_\zeta)$; e.g.

$$(1+z)^{-1}\propto\sum_{n=0}^\infty(-)^n z^n(E_\zeta)\text{ for }|\zeta=\arg z|<\pi.$$

In section 2 this theory is applied to the Γ-function (a.o. determination of general sharp upper bounds for the error term of $\log\Gamma(z)$).

In section 3 it is shown that under general conditions for each point z on the line segment $(0, w)$ (w real or complex $\neq 0$) in the Maclaurin expansion

$$f(z)=\sum_{h=0}^\infty \frac{f^{(h)}(0)}{h!}z^h+\frac{\lambda_n z^n}{n!},$$

λ_n belongs to a convex set E_n, assuming that $f^{(n)}(z)$ belongs to the same set. In section 4 the author obtains by integration by parts of $\int_a^b g(x)e^{i\omega f(x)}dx$ (a and b real) the expansion $\sum_{h=0}^{n-1}(\alpha_h+\beta_h)\omega^{-1-h}+\omega^{-n}I_n$, where the last term is under specified conditions $=O(|\omega|^{-n})$. The asymptotic sum of the asymptotically convergent series

$$\sum\alpha_h\omega^{-1-h}(h>0)\propto A$$

is called the asymptotic residue between a and a^+; notation $\int_a^{a+}g(x)e^{i\omega f(x)}dx$. In an analogous way

$$B\propto\sum\beta_h\omega^{-1-h}=\int_{b-}^{b}g(x)e^{i\omega f(x)}dx.$$

The author proposes to treat, in later reports, the complete theory of asymptotic residues. In section 5 absolutely monotonic and completely monotonic functions are treated. A function $f(x)$ is called absolutely monotonic in an interval j if $f^{(n)}(x)\geqq 0$ in j ($n=0, 1, 2, \cdots$) and completely monotonic in j if $(-1)^n f^{(n)}(x)\geqq 0$ in j ($n=0, 1, 2, \cdots$). For the restricted class of integrals containing such functions a very sharp numerical upper bound for the error term is obtained.

In the last chapter 6 sufficient conditions are obtained for

$$\lim_{h\to\infty}\sum_{0<n<m_h} f_n(h)=\sum_{n=1}^\infty\phi_n\ (m_h=\text{or }\to\infty\text{ for }h\to\infty),$$

where the last series converges (absolutely resp. unconditionally). Many applications are given (e.g. Γ-function, polynomials of Legendre, ultraspherical polynomials, formula of Laplace-Heine, values of derivatives of polynomials of Legendre at $x=\pm 1$, (formula of Grosswald, with generalisation), upper bound for the derivatives of the polynomials of Legendre (Picone), generalisation of the formula of Picone for ultraspherical polynomials, polynomials of Fejer, etc.). *S. C. van Veen.*

★**van der Corput, J. G. Asymptotic expansions. III. The asymptotic behaviour of the real solutions of certain second order differential equations.** Department of Mathematics, University of California, Berkeley, Calif., 1955. ii+171 pp. (mimeographed)

$Y(x)$ and $\eta(x)$ are two linearly independent solutions of

(1) $$y''+P(x)y'+Q(x)y=0.$$

It is possible to determine the asymptotic behaviour for large positive x of the real solutions of the differential

equation (2) $y''+\{P(x)+f(x)\}y'+\{Q(x)+g(x)\}y=0$ provided that $f(x)$ and $g(x)$ (perturbations) and their derivatives are sufficiently small for large positive x. (1) is called the related equation. In section 2 are introduced certain transformations, e.g. $y=r(x)(Y\sin\theta+\eta\cos\theta)$; $y'=r(x)(Y'\cos\theta+\eta'\cos\theta)$, by which (2) is transformed into (3) $\theta'=\Lambda(x,\theta)$; (4) $2r'r^{-1}=M(x,\theta)$. The advantage is that (3) is a differential equation of the first order not involving $r(x)$. After determination of the asymptotic behaviour of $\theta(x)$ (4) gives the asymptotic behavior of $2\log r(x)$, and therefore also $y(x)$ and $y'(x)$. Special case: $P(x)=0$, $Q(x)=1$, $f(x)$ and $g(x)$ absolutely integrable to infinity.

$$\theta(x)\to\gamma;\ 2\log r(x)\to\gamma^*\text{ for }x\to+\infty\ (\gamma\text{ and }\gamma^*\text{ finite}).$$

By putting $2\{\theta(x)-\gamma\}=\varepsilon(x)$ it is shown in section 4 that $\varepsilon(x)$ satisfies an integral equation of the form

$$\varepsilon(x)=\int_x^\infty\{\varkappa(t)+\lambda(t)\sin\varepsilon(t)+\mu(t)(1-\cos\varepsilon(t))\}dt$$

($\varkappa(x)$, $\lambda(x)$, $\mu(x)$ given functions, absolutely integrable to ∞). In order to write $\varepsilon(x)$ in a simple form an operation called bar-multiplication is introduced, defined as follows

$$j(x)=\int_{t_1\geq\cdots\geq t_m\geq x}\cdots\int f(x,t_1,\cdots,t_m)dt_1,\cdots dt_m;$$

$$J(x)=\int_{t_{m+1}\geq\cdots\geq t_{m+M}\geq x}\cdots\int F(x,t_{m+1},\cdots,t_{m+M})dt_{m+1}\cdots dt_{m+M}$$

The bar-product $j\cdot J$ is by definition the $m+M$-tuple integral:

$$\int_{t_1\geq\cdots\geq t_{m+M}\geq x}\cdots\int f(x_1,t_1,\cdots,t_m)F(x,t_{m+1},\cdots,t_{m+M})\,dt_1\cdots dt_{m+M}.$$

In section 4 it is shown that $\varepsilon(x)$ possesses an asymptotic expansion of the form

$$\varepsilon(x)\sim k+\overline{kl}+\overline{kll}+\overline{klkm}-\overline{kkll}+\cdots\ (k=\textstyle\int_x^\infty\varkappa(t)dt\text{ etc}).$$

The remainder is at most of the order ω^{-r} (r=the number of bar-factors in the first neglected term). An analogous result is obtained for $2\log r-\int_x^\infty f(t)dt-\gamma^*$.

In a purely formal manner the asymptotic expansion for

$$U(x)+iV(x)=\exp\{-\tfrac{1}{2}\textstyle\int_x^\infty f(t)dt-\frac{1}{2}\gamma^*)r\ \exp\{\frac{1}{2}i\varepsilon\}$$

($\frac{1}{2}\varepsilon=\theta-\gamma$)

is computed.

In this expansion certain regularities in the formation of the bar-products and in the signature of the terms lead to a conjecture, which is verified in section 4. In the more complicated case in which there is no absolute integrability, the author restricts himself in the sections 5 and 8 to differential equations of the form

$$5)\quad y''+y'\Big(\sum_{\nu=1}^{n}L_\nu f_\nu(x)\cos 2\alpha_\nu(x)+y\Big(1+\sum_{\nu=1}^{n}K_\nu f_\nu(x)\cos 2\alpha_\nu(x)\Big)=0,$$

f_ν and α_ν are given, real, infinitely differentiable functions, K_ν and L_ν are given constants. To determine the asymptotic behaviour of the real solution of $\theta'=\Lambda(x,\theta)$ a certain "smoothness condition" is imposed on the functions f_ν. On the functions α_ν a certain "resonance condition" is imposed which can be formulated roughly by saying that there is either complete resonance or no resonance at all. Under these two conditions it is possible to construct for each real solution $\theta(x)$ a real constant γ such that $\theta(x)-\gamma$ is for large positive x asymptotically equal to the asymptotic sum of an asymptotically convergent series. These two conditions however are not sufficient to deduce from this asymptotic equation an asymptotic expansion whose terms do not involve $\theta(x)$ itself. To that end a certain supplementary condition is needed (section 6). In section 8 the author demonstrates under these three conditions: I. Uniqueness theorem. If θ and θ^* denote two real solutions of $\theta'=\Lambda(x,\theta)$ with the same constant γ in their asymptotic equation, then $\theta=\theta^*$. II. Existence theorem. If γ denotes an arbitrary given real constant number, then it is possible to construct a real solution θ of $\theta'=\Lambda(x,\theta)$ such that γ is the constant term in the asymptotic equation of that solution. By I this solution is uniquely defined by γ.

In the sections 9 and 10 the asymptotic behaviour of the solutions of some more complicated differential equations is investigated. In section 11 the case is treated that in (5) f_ν and α_ν possess asymptotic expansions according to powers of x. In section 12 $\alpha_\nu(x)=\alpha_\nu x+\eta_\nu$ denotes a linear function of x. It is shown that the rationality properties of α_ν play an important role. In section 13 $n=1$, $\eta=0$, $f(x)$ is continuous of bounded variation. In this case however, there are some rational values which may not be assumed by α, e.g. the value zero. In the last section 14 the author obtains a first approximation to θ and $\log r$ under much weaker conditions. This section is closely related to stability theory. *S. C. van Veen* (Delft).

van der Corput, J. G. Asymptotics. IIIa. The asymptotic behaviour of the real solutions of certain second order differential equations. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 1–10.

van der Corput, J. G. Asymptotics. IIIb. The asymptotic behaviour of the real solutions of certain second order differential equations. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 11–14.

These two papers contain a summary of the introduction of the paper reviewed above. *S. C. van Veen.*

See also: Nagase, p. 1255.

Interpolation, Approximation, Orthogonal Functions

★**Erdélyi, A. Asymptotic expansions.** Dover Publications, Inc., New York, 1956. vi+108 pp. $1.35.

This book is the first published in English on the general topic of asymptotic expansions and their uses, which serves to indicate the growing interest in and the increase in value attached to this subject. The book is based upon a course of lectures intended to introduce students "to various methods for the asymptotic evaluation of integrals containing a large parameter and to the study of solutions of ordinary linear differential equations by means of asymptotic expansions." This aim is more than adequately achieved though the amount of space alloted to each topic is small.

The book consists of a short introduction and four chapters. References are given at the end of each chapter, but there is no attempt made at completeness. There is no index.

Chapter I: Asymptotic series. Definitions of asymptotic sequences, expansions, and representations are made. As defined, an asymptotic expansion may have any finite

number of terms, an asymptotic representation but one term. Operations with asymptotic expansions are discussed. Summation of asymptotic series is briefly mentioned. The subjects are developed only to the extent required for the sequel.

Chapter II: Integrals. It is shown how asymptotic expansions may be obtained by repeated integrations by parts, the method of steepest descents, and the method of stationary phase. Several examples are given, including a discussion of Laplace and Fourier integrals.

Chapter III: Singularities of differential equations. Here a brief introduction to the theory of asymptotic solution of ordinary homogeneous linear differential equations of the second order is to be found. Presentations of the existing theory of this and the first topic of the final chapter can be found in other texts. It is shown that segments of normal and subnormal solutions are asymptotic expansions of true solutions. The Stokes' phenomenon is discussed. Application is made to the Bessel functions of order zero and large argument.

Chapter IV: Differential equations with a large parameter. The asymptotic solutions with respect to the complex parameter λ of $y''+q(x,\lambda)y=0$, wherein $q(x,\lambda)=\sum_0^\infty q_n(x)\lambda^{2k-n}$, k being a positive integer, and $q_0(x)$ nonzero in the considered interval of x, are determined. This is the so-called classical case. The theory is illustrated by a brief investigation of Bessel functions of large complex order. Lastly, a segment of the theory of asymptotic solution of differential equations with transition (turning) points is presented. In particular, solutions of $y''+[\lambda^2 p(x)+q(x)]y=0$ over an interval containing a simple zero of $p(x)$ are considered. Application is made to obtain uniform asymptotic representations of Bessel functions of large complex order. Some simple examples of divergent and convergent formal solutions of equations of the form $y''+q(x,\lambda)y=0$ could well have been included. As it is, here and in Chapter III, it is only stated that the formal solutions are in general divergent.

The discussions are lucid and well-motivated, and the reader is encouraged to step beyond the stopping points in the text. The need for a book covering the above topics has long been felt, and the author has made just the proper selection from a large accumulation of material in the field of asymptotic expansions necessary to create a true and thorough introduction. *N. D. Kazarinoff.*

Badalyan, G. V. Generalization of Legendre polynomials and certain applications of them. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **9** (1956), no. 1, 3–22. (Russian)

This is the second part of a paper whose first installment has already been reviewed [same Izv. **8** (1955), no. 5, 1–28; MR **17**, 844]. The first section of this part deals with functions of the class H_1^2, i.e. those which are Laplace transforms of functions ϕ such that $e^{-x/2}\phi \in L^2(0,\infty)$. The author introduces $\Phi_n(z)$, the Laplace transforms of the generalized Legendre polynomials defined in the first part, and shows that if $\sum(\gamma_\nu+1)^{-1}<\infty$ then $\{\Phi_n(z)\}$ is fundamental in H_1^2 for $\Re z>\frac{1}{2}$. He gives many other results on approximation and extremal problems involving H_1^2 and the Φ_n. The next section deals with W'_σ, W''_σ, the classes of finite Fourier transforms (over $(-\sigma,\sigma)$) of odd or even functions of L^2. An element f of W_σ'' belongs to $W_\sigma''\{\mu_{n_\nu},0\}$ if $f^{(n_\nu)}(0)=\mu_{n_\nu}$ $(k=0,1,\cdots,p)$. The author determines $\inf\int_{-\infty}^{\infty}|f(u)|^2du$ in this class, and similarly for some other classes. He also obtains an inequality for the best quadratic approximation to a given function by linear combinations of $\{x^{n_\nu}\}$. The last section deals with a generalization of Chebyshev polynomials: these are those linear combinations of the functions $\omega_k(x)$ (introduced in the first part) which deviate least from zero. Some inequalities are given for them. *R. P. Boas, Jr.*

Geronimus, Ya. L. On the closure of certain systems of functions in L_σ^p space. Har'kov. Gos. Univ. Uč. Zap. **29**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **21** (1949), 23–45. (Russian)

Proofs of results announced in Dokl. Akad. Nauk SSSR (N.S.) **62** (1948), 9–12; **64** (1949), 611–614 [MR **10**, 295, 530]. *W. H. J. Fuchs* (Ithaca, N. Y.).

Endl, Kurt. Über eine ausgezeichnete Eigenschaft der Koeffizientenmatrizen des Laguerreschen und des Hermiteschen Polynomsystems. Math. Z. **65** (1956), 7–15.

Proofs are supplied for the results stated in an abstract [C.R. Acad. Sci. Paris **241** (1955), 682–684; MR **17**, 149]. In addition, the polynomials $P_{k,n}(z)$ orthogonal over $\{re^{2\pi il/k};\ 0\leq r\leq R\leq\infty,\ l=0,1,2,\cdots,k-1\}$ with respect to the weight function $\exp(-|z|^k)$ are shown to be involutive, as defined in that abstract, for $k=1,2,3,\cdots$. *A. E. Livingston* (Seattle, Wash.).

Salzer, Herbert E. Orthogonal polynomials arising in the numerical evaluation of inverse Laplace transforms. Math. Tables Aids Comput. **9** (1955), 164–177.

In finding the inverse Laplace transform $f(t)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}e^{pt}F(p)dp$ of $F(p)$ the function $F(p)$ may be either known only numerically or too complicated for evaluating $f(t)$ by Cauchy's theorem.

The purpose of the author is to discuss the properties of a new set of orthogonal polynomials which can be the basis for convenient formulas for approximating $f(t)$ for different positive values of t.

An explicit expression for these polynomials is given by

$$P_n\Big(\frac{1}{p}\Big)=(-1)^n e^{-p}p^n\frac{d^n}{dp^n}(e^p p^{-n})$$

$$=(-1)^n\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\binom{2n-r-1}{n-r}(n-r)!\,p^{r-n}$$

$$=(-1)^n\sum_{r=0}^{n}(-1)^r\frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-\overline{r-1}^2)}{r!}p^{-r}$$

(with integral coefficients). Recurrence formula:

$$(2n-3)P_n(x)=[(4n-2)(2n-3)x+2]P_{n-1}(x)+(2n-1)P_{n-2}(x)\quad(n\geq 3).$$

Differential equation:

$$x^2P_n''(x)+(x-1)P_n'(x)-n^2P_n(x)=0.$$

Explicit expressions for $P_n(x)$ are given for $n=1(1)12$.

In a numerical table there are given the values of the reciprocals of the zeros of $P_n(x)$ or $p_i^{(n)}$, the zeros of $P_n(x)$ or $1/p_i^{(n)}$, and the corresponding Christoffel numbers $A_i^{(n)}$, for $n=1(1)8$. *S. C. van Veen* (Delft).

Endl, Kurt. Orthogonalisierung auf einem k-strahligen symmetrischen Integrationsstern. Math. Z. **65** (1956), 1–6.

If

$$S\equiv S_{k,R}=\{re^{2\pi il/k};\ 0\leq r\leq R\leq\infty,\ l=0,1,2,\cdots,k-1\}$$

and if $T(z)\equiv T_k(z)=ze^{2\pi i/k}$, then $T(S)=S$. Consequently,

if w is a weight-function for S with the property that $w \circ T = w$ and if $P_n(z) = z^n + \alpha_{n1} z^{n-1} + \cdots + \alpha_{nn}$ $(n = 0, 1, 2, \cdots)$ is the set of orthogonal polynomials obtained by orthogonalizing $\{z^n\}_{n=0}^{\infty}$ by the usual Gram-Schmidt procedure with the inner product

$$(u, v) = -i \int_S u(z)\overline{v(z)} z^{-1} w(z)\, dz,$$

then standard arguments show immediately that $P_n \circ T = e^{2\pi i n/k} P_n$ and, hence, that $\alpha_{nj} = 0$ for $j \not\equiv n \pmod{k}$. In this paper, the above result is obtained by an induction argument on the degree of P_n. The author proves two further obvious results relating the polynomials $P_n(z)$ to certain polynomials orthogonal on a single ray of S with respect to a special weight function peculiar to that ray.

A. E. Livingston (Seattle, Wash.).

March, N. H.; and Plaskett, J. S. The relation between the Wentzel-Kramers-Brillouin and the Thomas-Fermi approximations. Proc. Roy. Soc. London. Ser. A. **235** (1956), 419–431.

The approximation in obtaining the Thomas-Fermi (TF) formula starting from the Wentzel-Kramers-Brillouin (WKB) method is analysed on some illustrative examples. Possible ways of correcting the errors introduced by integrating instead of summing are discussed. A modified formula for atomic binding energy has been obtained as a result of this analysis. *L. Infeld.*

★ **Exner, G.; und Hirschleber, A. Auswahlordinatenverfahren mit Planckscher Funktion als Basis.** Jenaer Jahrbuch 1955, 1. Teil, pp. 113–125. Gustav Fischer Verlag, Jena, 1955.

The authors are interested in approximate evaluation of integrals of the form $\int_0^\infty E_s(\lambda, T_1) f(\lambda) d\lambda$, where $E_s(\lambda, T_1)$ is the spectral radiancy of a black body of temperature T_1 at wave length λ. To this end they recommend choosing values $\lambda = \lambda_{ri}$ scattered through regions containing fractions $9 \cdot 10^{-r}$ of the total radiation and replacing the integral by an approximate sum

$$F \sim 1.646 T_1^4 \sum_{r=1}^{\infty} \left[\frac{10^{-r}}{N_r} \sum_{i=1}^{N_r} f(\lambda_{ri}) \cdot 10^{-4} \right.$$

They list suggested values to approximately 3S for λ_{ri}, $r = 1(1)9$ $i = 1(1)N_r$, $N_1 = 10$, $N_2 = N_3 = N_4 = N_5 = 5$, $N_6 = 2$, $N_7 = N_8 = N_9 = 1$.

An auxiliary table extracted from earlier tables by Lowan and Blanch [J. Opt. Soc. Amer. **30** (1940), 70–81; MR **1**, 252] is reproduced to help in the determination of λ_{ri}. Several examples are shown. *C. B. Tompkins.*

Wagner, H. Eine Näherungsformel für die Schwingungsdauer eines Pendels. Elem. Math. **11** (1956), 82–86.

See also: Makarov, p. 1255.

Trigonometric Series and Integrals

Mohanty, R.; and Nanda, M. On the jump of a function and its Fourier coefficients. Bull. Calcutta Math. Soc. **47** (1955), 157–163.

Let $\sum B_n(x)$ be the conjugate series of the Fourier series of the Lebesgue integrable function $f(t)$. Let $\theta(t) = f(x+t) - f(x-t)$ and $\psi(t) = \theta(t) - d(x)$. If for $\delta > 0$,

$$\int_h^\delta \left| \frac{\psi(t)}{t} - \frac{\psi(t+h)}{(t+h)} \right| dt \to 0 \text{ as } h \to 0+,$$

then the sequence $\{nB_n(x)\}$ is $(C, 1)$ summable to $d(x)/\pi$. If $\int_0^\delta \theta(t) \cot \frac{1}{2}t\, dt$ exists as a Cauchy integral and if

$$\int_h^\delta |(t+h) - (t)|/t\, dt \to 0 \text{ as } h \to 0+,$$

then $\sum B_n(x)$ is convergent. *P. Civin.*

Mulholland, H. P. On two extremum problems for polynomials on the unit circle. J. London Math. Soc. **31** (1956), 191–199.

Let $\mathfrak{P}_n$ denote the class of complex polynomials $p(z) = \sum_{k=0}^n c_k z^k$ of degree at most n, with c_0 real and $|\Re\, p(z)| \leq 1$ when $|z| \leq 1$. The author solves extremum problems in which Rogosinski's general results [same J. **29** (1954), 259–275; MR **16**, 29] are as helpful as possible. The first is to find $C_k(n) = \max |c_k|$: he finds $C_k(n) = 2l^{-1} \cot(\frac{1}{2}\pi/l)$, where $l = [\frac{1}{2}n/k + 3/2]$. The second [previously solved by other methods by Szegö, Amer. J. Math. **65** (1943), 532–536; MR **5**, 65] is to find $\max |\Im\, p(z)|$.

R. P. Boas, Jr. (Evanston, Ill.).

Integral Transforms, Operational Calculus

Rumšiskiĭ, L. Z. The Laplace transform and positive functions. Har'kov. Gos. Univ. Uč. Zap. **29** = Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **21** (1949), 101–130. (Russian)

Let P_A and P_A' $(0 < A \leq \infty)$ be the classes of continuous functions defined on $(-A, A)$ and which have there the representations

$$M(t) = \int_{-\infty}^{\infty} e^{itu} d\sigma(u) \text{ and } F(t) = \int_{-\infty}^{\infty} e^{itu} \phi(u) du$$

respectively, where $d\sigma(u) \geq 0$ and bounded and $0 \leq \phi(u) \leq 1$. If $L\{M; z\} = \int_0^\infty e^{-zt} M(t)\, dt$ $(x = \text{Re}\,(z) > 0)$, then it is well-known that $M(t) \in P_\infty$ $(M(t) \not\equiv 0)$ if and only if $\text{Re}(L\{M; z\}) > 0$ and $xL\{M; x\}$ is bounded for $x > 0$. The author shows that $F(t) \in P_\infty'$ if and only if $0 < \text{Re}\,(L\{M; z\}) < \pi$ and $xL\{F; x\}$ is bounded for $x > 0$. Similar theorems are proved for functions which may be represented as $\int_a^b e^{itu} d\sigma(u)$ or $\int_a^b e^{itu} \phi(u)\, du$, where a and b are finite and $d\sigma(u)$ and $\phi(u)$ satisfy the above conditions.

The author also considers the problem of the relationship between the classes P_A and P_A'. Let $L_A\{M; z\} = \int_0^A e^{-zt} M(t)\, dt$. The following theorem is proved: Let $F(t)$ be a continuous and Hermitian $(F(t) = \overline{F}(-t))$ function defined on $(-A, A)$. A necessary and sufficient condition that $F \in P_A'$ is that

$$\exp(iL_A\{F; z\}) - 1 = iL_A\{M; z\} + g(z) \quad (\text{Re}\,(z) > 0),$$

where $M(t) \in P_A$ and the function $g(z)$ is regular in $\text{Re}(z) > 0$ and in some half-plane $\text{Re}(z) \geq \delta > 0$ has the estimate $g(z) = o(e^{-Ax})$ for $z \to \infty$. In the case where $A = \infty$, $g(z) \equiv 0$ and in this case the author shows that to prove the sufficiency part of this theorem it is enough to assume the above relationship for only a sequence $z_0 + n\lambda$ where $\text{Re}(z_0) > 0$, $\lambda > 0$ and $n = 0, 1, \cdots$. The proof of this fact follows very closely the proof of the analogous fact in the so called L-problem of moments.

Finally, suppose that $F_1 * F_2(t) = \int_0^t F_1(t-u) F_2(u)\, du$ and that $F^{*n}(t)$ is this convolution of F taken with itself n times. Then the following is proved: There is a one to one relationship between the functions $F(t)$ of the class P_A' and the functions $M(t)$ of the class P_A given by the formula

$$M(t) = F(t) + \sum_{n=2}^{\infty} \frac{i^{n-1}}{n!} F^{*n}(t)$$

or inversly

$$F(t)=M(t)+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}M^{*n}(t).$$

A. Devinatz (St. Louis, Mo.).

Povzner, A. Ya. On expansions in functions which are solutions of a scattering problem. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 360–363. (Russian)

Consider the differential operator $A=-\Delta+q(x)$ in real three-space R^3. When q is bounded from below, it is self-adjoint on $H=L^2(R^3)$. When $q=0$, it is diagonalized by the Fourier transform $(Tf)(\xi)=\int e_0(\xi,x)f(x)dx$, $(e_0=(2\pi)^{-3/2}\exp -i\xi x)$, which is unitary from H to H. It is shown that when q vanishes outside a compact set and A has no negative spectrum (which is necessarily discrete), then the analogous diagonalizing transform is $(Tf)(\xi)=\int e(\xi,x)f(x)dx$, where e is the unique solution of $Ae=|\xi|^2e$, $(|\xi|^2=\xi_1^2+\xi_2^2+\xi_3^2)$, of the form $e=e_0+e_1$ where $e_1=O(|x|^{-1})$ and $|x|((de_1/d|x|)-i|\xi|e_1)=O(|x|^{-1})$ as $|x|\to\infty$. As before, T is unitary from H to H. If A has negative spectrum, T diagonalizes the positive part of A.

L. Gårding (Lund).

Wolf, Frantisek. Perturbation by changes of one-dimensional boundary conditions. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 360–366.

Let A, B be two self adjoint operators in a Hilbert space which are related as follows: there exists an operator A' which is a restriction of A as well as B and is such that the quotient spaces $D(A)/D(A')$ and $D(B)/D(A')$ have both dimension 1 if $D(\ )$ denotes the domain of the operator in question. In the concrete situation referred to in the title of the paper, A is a differential operator with boundary conditions and B is obtained from A by replacing a "one dimensional" boundary condition U of A by a one dimensional boundary condition V while A' is the operator obtained from A by adding the boundary condition V. (The case that A and B differ by a finite number of boundary conditions can be treated by iteration). The paper deals with the relation of the spectrum of B to that of A and has points of contact with a paper by Friedrichs [Comm. Appl. Math. **1** (1948), 361–406; MR **10**, 547].

Under the condition that the resolvent sets of A and B have at least one real point μ in common it is first proved that there exists a real γ and an element x_0 of the Hilbert space such that the perturbing operator

$$P(x)=((\mu-A)^{-1}-(\mu-B)^{-1})(x)$$

equals $x_0(x,x_0)/\gamma$. By a suitable decomposition of the Hilbert space it is further shown that the perturbation problem can be reduced to the case of an operator with a simple spectrum. Due to a general theorem in Hilbert space theory it can then be reduced to the case where the Hilbert space is a certain L^2 space of function $g(x)$ and the operation A consists in multiplying $g(x)$ by x.

Among the various results of the paper concerning the spectra of A and B we mention the following one: if z is an m-tuple eigenvalue of A then z is an eigenvalue of B of multiplicity $m-1$ or m, and it can be of multiplicity $m+1$ only if a certain relation $f(z)=0$ involving x_0, γ, and the spectral decomposition of A is satisfied. *E. H. Rothe.*

See also: Salzer, p. 1203; Altman, p. 1226; Moisil, p. 1248; Dengler, p. 1254; Petrašen', p. 1255; Satô, p. 1255; Gel'fand and Yaglom, p. 1261; Turner, p. 1261.

Special Functions

Tricomi, Francesco G. Cosa sono e a che servono le funzioni ipergeometriche confluenti. Rend. Sem. Mat. Fis. Milano **25** (1953–54), 3–14 (1955).

In this expository lecture the author gives an introduction to confluent hypergeometric functions and emphasizes his preference for the notation used in his books and other writings on the subject as compared with the notation introduced by E. T. Whittaker and used by Buchholz and some other writers. *A. Erdélyi.*

Smith, R. C. T. Generating functions of Appell form for the classical orthogonal polynomials. Proc. Amer. Math. Soc. **7** (1956), 636–641.

A sequence of polynomials $\{P_n(x)\}$ is said to be a generalized Appell set if they have a generating function of the form

$$K(x,w)=\sum_{0}^{\infty}w^nP_n(x)=A(g(w))\psi(xg(w))$$

where $\psi(t)=1+\psi_1t+\cdots$, $A(g)=1+a_1g+\cdots$, $g(w)=g_1w+g_2w^2+\cdots$. If the polynomials satisfy a differential-difference equation in n and x, then it follows that K must satisfy a related partial differential equation in x and w. The author uses this to show that the Jacobi polynomials $P_n^{(\alpha,\beta)}$ are of Appell type only when $\alpha-\beta$ has one of the values 1, 0, −1. [The reviewer wishes to point out that this does not give a complete answer to the problem which he originally raised since, for example, the polynomials $P_n^{(\alpha,\beta)}(x-1)$ are of Appell type for any choice of α and β.] *R. C. Buck* (Madison, Wis.).

Carlitz, L. Some polynomials related to theta functions. Ann. Mat. Pura Appl. (4) **41** (1956), 359–373.

The author studies the polynomials

$$H_n(x,q)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}x^r,\quad G_n(x,q)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}q^{r(r-n)}x^r$$

introduced by Szegö. Here $|q|<1$,

$$\begin{bmatrix}n\\0\end{bmatrix}=1,\quad\begin{bmatrix}n\\r\end{bmatrix}=\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}.$$

They have formal properties which are analogous to corresponding properties of Hermite polynomials.

A. Erdélyi (Pasadena, Calif.).

Conte, S. D. The operational calculus of Gegenbauer transforms. Z. Angew. Math. Mech. **36** (1956), 148–150.

The Gegenbauer transform of $F(x)$ is

$$\int_{-1}^{1}F(x)C_n^{\nu}(x)(1-x^2)^{\nu-\frac{1}{2}}dx\quad(n=0,1,2,\cdots),$$

i.e., virtually the sequence of coefficients in the expansion of $F(x)$ in a series of Gegenbauer polynomials $C_n^{\nu}(x)$. The author obtains some formal properties of this transformation including a convolution formula. *A. Erdélyi.*

Cheng, David, K. Solution of an integral. Elec. Engrg. **75** (1956), 673.

The author obtains

$$\int_0^{\pi/2}\operatorname{cosec}\phi J_0(u\sin\phi)\cos^2\left(\frac{\pi}{2}\cos\phi\right)d\phi$$

$$=\frac{\pi^2}{2}\sum_{k=0}^{\infty}(-1)^kA_ku^{-k}\sqrt{\left(\frac{\pi}{2u}\right)}J_{k+\frac{1}{2}}(u),$$

where

$$A_k=\frac{\pi^{2k}}{2^k k!}\sum_{i=0}^{\infty}(-1)^i\frac{\pi^{2i}}{(2i+2k+1)(2i+1)!}.$$

A. Erdélyi (Pasadena, Calif.).

Toscano, Letterio. Teorema di moltiplicazione sulle funzioni ipergeometriche generalizzate. Ann. Mat. Pura Appl. (4) **41** (1956), 269–288.

The principal result of this paper is

$${}_pF_q(a_1, \cdots, a_p; c_1, \cdots, c_q; kx)=$$

$$\sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_p)_m(-x)^m}{(c_1)_m\cdots(c_{q-1})_m(\nu+m-1)_m}\,{}_2F_1(-m, \nu+m-1; c_q; k)$$

$$\times\,{}_pF_q(a_1+m, \cdots, a_p+m; c_1+m, \cdots, c_{q-1}+m, \nu+2m; x).$$

There are many special cases and applications of this result. *A. Erdélyi* (Pasadena, Calif.).

See also: Erdélyi, p. 1202; Petrašen', p. 1255.

Ordinary Differential Equations

★ **Birger, I. A. Nekotorye matematičeskie metody rešeniya inženernyh zadač. [Some mathematical methods for solving engineering problems.]** Gosudarstv. Izdat. Oboronnoĭ Promyšlennosti, Moscow, 1956. 151 pp. 5.70 rubles.

This book presents methods of solving linear ordinary differential equations and integral equations in a form suitable for applied problems. These methods are for the most part different from those found in existing books addressed to engineers.

Let the general linear differential equation of nth order be (*) $L(y)=f(x)$ and let the coefficients first be constants. A solution is sought in the interval $a\leqq x\leqq b$. That fundamental set $\{Y_k(x)\}$ of solutions of the homogeneous equation for which $Y_k^{(i)}(a)=\delta_k^i$ ($i, k=0, \cdots, n-1$) is called the normal fundamental set. It is shown how the general solution of (*) can be constructed from a knowledge of $Y_{n-1}(x)$ alone, which in turn is calculated from the roots of the characteristic polynomial. The procedure is extended to solutions satisfying initial conditions having prescribed discontinuities of first order in the derivatives through the $(n-1)$st. When the coefficients are functions of the independent variable the normal fundamental set must be determined approximately. For this purpose the boundary-value problem is converted to an integral equation, for which three methods of solution are given.

A chapter on integral equations per se follows. Two types are treated, which are modifications of the Fredholm and Volterra equations, respectively; the classical solutions themselves, however, are rejected as not useful for applications. Methods of the iterative type for the approximate solution of homogeneous and non-homogeneous cases of each are given. Criteria of convergence and applicability are discussed. Many non-trivial examples, chosen mainly from structural mechanics, illustrate the methods and constitute an important part of the book. *R. N. Goss* (San Diego, Calif.).

Turrittin, Hugh L. Rappresentazione mediante serie delle soluzioni delle equazioni differenziali ordinarie lineari. Confer. Sem. Mat. Univ. Bari no. **17** (1956), 16 pp.

This is an expository article giving a lucid account of series solutions of equations of the type $t^h dX/dt=A(t)X$ ($h=0, 1, \cdots$), where X and A are $n\times n$ matrices and the elements of A are holomorphic when $|t|<R_0$. The author limits himself to the representative case $n=2$. The character of series expansions of solutions when $h=0$, 1, or 2 is discussed. The use of Laplace transforms in obtaining convergent series from divergent ones and the topic of factorial series are touched upon. The (unsettled) question of how to find connection formulas relating solutions at $t=0$ with those at $t=\infty$ is mentioned. There is a bibliography of twenty-one items. *N. D. Kazarinoff.*

Bellman, R.; Glicksberg, I.; and Gross, O. On the "bang-bang" control problem. Quart. Appl. Math. **14** (1956), 11–18.

The state of a physical system is described by an n-dimensional vector function $z(t)$ which satisfies

$$dz/dt=Az+f,\ z(0)=c.$$

A is a real constant matrix. The "forcing term" f is restricted to real, measurable functions with components f_i satisfying $|f_i|\leqq 1$. In a control system $z(t)$ is the transient error and the restriction $|f_i|\leqq 1$ corresponds to limited power available to the feedback network. The control problem is to select an f that reduces $z(t)$ to zero in minimum time.

Under the assumption that the characteristic roots are negative and distinct, it is shown that a minimizing function f exists with $|f_i|=1$ and with each f_i changing signs at most $(n-1)$ times. The elegant proof makes use of properties of convex sets. The authors point out that a similar proof establishes that, if the characteristic roots of A have negative real parts, there exists a minimizing function f with $|f_i|=1$. This means that the transient error is reduced most rapidly by using properly all the power available. A method for determining where in the phase plane switching (of signs) should occur is illustrated by a specific 2-dimensional example. *J. P. LaSalle.*

Gagliardo, Emilio. Teoremi di unicità per le soluzioni di un'equazione differenziale ordinaria del primo ordine. Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 160–171 (1956).

In this paper several different concepts of a solution of the equation $y'=f(x, y)$ are considered, e.g.: (1) an absolutely continuous function satisfying the equation almost everywhere; (2) an absolutely continuous function having a right-hand derivative everywhere and satisfying the equation everywhere when y' is interpreted as the right-hand derivative; (3) the same as (2) except that the equation need be satisfied only almost everywhere. With respect to these various concepts of a solution, the author discusses critically several theorems concerning the uniqueness of solutions and generalizing the classical Lipschitz criterion. The theorems, which are mostly improvements of previously known theorems, are too complicated to be stated in a review. *L. A. MacColl.*

★ **Erdélyi, A. Asymptotic factorization of ordinary linear differential operators containing a large parameter.** Tech. Rep. 8. Department of Mathematics, California Institute of Technology, Pasadena, 1956. 28 pp.

The author considers the differential equation

$$Ly=D^n y+\lambda^{\varrho}p_1(x, \lambda)D^{n-1}y+$$

$$\lambda^{2\varrho}p_2(x, \lambda)D^{n-2}y+\cdots+\lambda^{n\varrho}p_n(x, \lambda)y=0$$

for $\varrho>0$ and integral, where each $p_i(x, \lambda)$ has an asymp-

totic expansion of the form

$$p_i(x,\lambda)\sim\sum_{k=0}^{m-1}p_{ik}(x)\lambda^{-\sigma k}+O(\lambda^{-\sigma m}) \text{ as } \lambda\to\infty,$$

for all m up to some fixed, but arbitrary N. All asymptotic representations are taken in this sense. The $p_{ik}(x)$ are assumed sufficiently smooth so that as many derivatives as required exist.

The characteristic polynomial of $Ly=0$ is

$$\Omega(x,\omega)\equiv\omega^n(x)+p_{10}(x)\omega^{n-1}(x)+\cdots+p_{n0}(x)=0,$$

and it is shown that if $\Omega(x,\omega)$ is factored into relatively prime factors

$$\Omega(x,\omega)=\Omega_1(x,\omega)\cdots\Omega_H(x,\omega),$$

then to each $\Omega_k(x,\omega)$ corresponds a differential operator L_k with leading coefficient unity determined to within a differential operator of the form $\lambda^{-N}K$, where

$$K=\lambda^{\sigma}f_1(x,\lambda)D^{n-1}+\cdots\lambda^{n\sigma}f_n(x,\lambda)$$

and $\Omega_k(x,\omega)$ is the characteristic polynomial of L_k. If M is the least common multiple of the L_k, then $M-L=\lambda^{-N}K$, and every solution of $Ly=0$ is asymptotically representable by some solution of $My=0$. All that is actually needed is the asymptotic development of the solutions of $L_k=0$.

The current status of existing types of asymptotic expansions for particular forms of $L_ky=0$ is surveyed, giving the results for distinct linear factors of the characteristic polynomial, for repeated roots, and for simple transition points pertaining to quadratic factors, as special applications of the main theorem, which is a generalization of a result due to R. E. Langer.

G. Latta (Stanford, Calif.).

Moiseev, N. N. On the theory of asymptotic representations of integrals of linear differential equations having a parameter. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **32** (1955), no. 4, 131–134. (Russian)

The author considers the second order differential equation

$$y''+\lambda^2\Big[a^2(t)+\frac{b(t,\lambda)}{\lambda}\Big]y=0$$

on $0\leqq t\leqq T$ for $\lambda\to\infty$. Here $b(t,\lambda)$ is uniformly bounded. In case $b\equiv0$ the resulting equation has independent solutions u and v such that

$$u=a^{-\frac{1}{2}}(t)\cos(\lambda\int_0^t a(x)dx)+O(\lambda^{-1})$$

and similarly for v with cos replaced by sin. It is then shown that the differential equation containing b has a solution of the form

$$y=Cu(t)\cos\xi+Cv(t)\sin\xi+Z(\lambda,t)\lambda^{-1},$$

where $\xi=\int_0^t b/(2a)dt+\eta$ and C and η are constants and Z is uniformly bounded. *N. Levinson* (Cambridge, Mass.).

Olver, F. W. J. Errors in asymptotic solutions of linear ordinary differential equations. Quart. Appl. Math. **14** (1956), 218–219.

It is shown that the chain of argument in the paper of the same title by R. L. Evans [same Quart. **12** (1954), 295–300; MR **16**, 131] has flaws. It is also stated that the conclusions in that paper are in fact inconsistent. In the reviewer's opinion, the conclusions referred to by the author and those of Evans may not coincide. The former are unreasonable; the latter are not demonstrated.

N. D. Kazarinoff (Ann Arbor, Mich.).

Borghi, D. C. General solutions of second order linear homogeneous differential equations. Ann. Geofis. **8** (1955), 167–180.

Voir le rapport de Čakalov [Ref. Ž. Mat. **1956**, no. 4462], où l'on montre que le résultat de cette Note n'est pas exact. *D. S. Mitrinovitch* (Belgrade).

Zbornik, Josef. Zur Auflösung linearer homogener Differentialgleichungen 2. Ordnung. Z. Angew. Math. Phys. **7** (1956), 64–74.

L'auteur donne un procédé de formation d'équations différentielles linéaires, intégrables par des quadratures. Les résultats indiqués peuvent être employés pour combler des lacunes dans le Recueil d'équations différentielles inséré dans le Traité de Kamke [Differentialgleichungen, Lösungsmethoden und Lösungen, Bd. 1, 3. Aufl., Akademische Verlagsgesellschaft, Leipzig, 1944; MR **9**, 33]. *D. S. Mitrinovitch* (Belgrade).

Seifert, George. On stability questions for pendulum-type equations. Z. Angew. Math. Phys. **7** (1956), 238–247.

Let $y(\theta)y'(\theta)=g(\theta)-f(\theta,\alpha)y(\theta)$, where $\alpha>0$ is a parameter, f, g are continuously differentiable and periodic with period 2π in θ, f is continuous in α, and g has only simple zeros. The author shows that under further conditions on f there exist constants $\alpha_2>\alpha_1>0$ such that the positive limit set of every trajectory is (i) for any $\alpha>\alpha_1$, either a point or a closed curve, (ii) for any $\alpha>\alpha_2$, a point; and that under certain conditions on f and g there exists an $\alpha_0>0$ such that, for any positive $\alpha<\alpha_0$, (i) does not hold. The case $f(\theta,\alpha)\equiv\alpha f(\theta)$, $f(\theta)>0$, has been previously discussed by the author jointly with J. C. Lillo [same Z. **6** (1955), 239–243; MR **16**, 1025]. *H. A. Antosiewicz.*

Grensted, P. E. W. Stability criteria for linear equations with time-varying coefficients. J. Roy. Aero. Soc. **60** (1956), 205–208.

The author proposes an admittedly non-rigorous method for obtaining criteria, similar to the Routh-Hurwitz criterion, for the exponential-asymptotic stability of the trivial solution of linear homogeneous differential equations with variable coefficients. The criterion for second order equations agrees with a theorem of Sonin [Zap. Imp. Akad. Nauk, **69** (1892), 1–30; cf., e.g., G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ. v. 23, New York, 1939, p. 161; MR **1**, 14].

H. A. Antosiewicz (Washington, D.C.).

Razumihin, B. S. On stability of unsteady motion. Prikl. Mat. Meh. **20** (1956), 266–270. (Russian)

The author reviews methods — said to be due to N. G. Četaev — for studying the stability of the unperturbed motion of a linear system. In one case the coefficients are nearly constant and in the other case the coefficients are slowly varying. The methods are then generalized to apply to the linear system

$$dx_i/dt=\sum_{j=1}^{n}p_{ij}(t)x_j \qquad (i=1,\cdots,n)$$

under the assumption that each $p_{ij}(t)$ and its derivative are bounded for $t>0$ and that the characteristic roots $\lambda_i(t)$ of the matrix $(p_{ij}(t))$ each satisfy $\mathrm{Re}(\lambda_i(t))<-\delta$ for some $\delta>0$ and all $t>0$. *J. P. LaSalle.*

Mitrinovitch, Dragoslav S. Sur l'équation différentielle de Sommerfeld pour la stabilité hydrodynamique. C. R. Acad. Sci. Paris **242** (1956), 2287–2289.

The author suggests a new approach to the problem of solving the differential equation

$$1) \quad (u-c)(\varphi''-\alpha^2\varphi)-u''\varphi+\frac{i}{\alpha R}(\varphi^{(4)}-2\alpha^2\varphi''+\alpha^4\varphi)=0$$

in which φ is an unknown function of the independent variable y; u is a given function of y, and α, c, R are parameters. It is proposed that (1) be treated as an underdetermined differential equation in the two unknown functions φ and u. The author indicates a procedure which permits the calculation of an infinite sequence of pairs of solutions $\{\varphi_k(y), u_k(y)\}$ by means of quadratures. It is not stated how this result could be used to solve (1), when u is a prescribed function. *W. Wasow* (Los Angeles, Calif.).

Utz, W. R. Su una nota di De Castro. Boll. Un. Mat. Ital. (3) **11** (1956), 28–30.

De Castro [same Boll. (3) **9** (1954), 167–169; MR **16**, 250] has stated a theorem according to which the periodic solution of the differential equation $\ddot{x}+g'(x)\dot{x}+f(x)=e(t)$ with $e(t+T)=e(t)$ is unique and stable under certain conditions. The author points out an error in De Castro's proof without, however, settling the question as to the validity of the theorem. The second part of the paper contains proofs of two theorems stating sufficient conditions for the boundedness of all solutions of the same differential equation. The proofs employ a technique of Urabe [Math. Japon. **2** (1950), 23–26; MR **12**, 182].

W. Wasow (Los Angeles, Calif.).

★ **Huang, T. C. Subharmonic oscillations in nonlinear systems of two degrees of freedom.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 95–100. American Society of Mechanical Engineers, New York, 1955. $9.00.

In spite of the title, one type of subharmonic oscillation in one system is considered. This is the oscillation of frequency one-third the driving frequency in a coupled mechanical system containing the following in a straight line: a fixed anchor, a linear spring, a mass capable is sliding along the line without friction, a slightly nonlinear spring whose nonlinearity is cubic, a second mass similar to the first, and a sinusoidal driving force directed along the line. The system is treated formally, oscillations of assumed frequency and unknown coefficients being introduced into the equations. When all oscillations of frequency other than those assumed are ignored, a system of algebraic equations determines the unknown coefficients. The problem of obtaining a practical solution to this solution is solved in this case by an iteration process based on the case of zero nonlinearity. The analysis is repeated when the spring between the two masses is accompanied by a small linear velocity damper.

E. Pinney (Berkeley, Calif.).

Leonov, M. Ya. Approximate method of investigating quasi-harmonic oscillations. Inst. Mašinoved. Avtomat. Nauč. Zap. **2** (1953), no. 1, 5–8. (Russian)

By a change in the independent variable the solution of a class of 2nd order linear differential equations is reduced to the problem of solving a first order nonlinear differential equation. Approximate solutions of the second order equation are obtained from approximate solutions of the first order equation. [The first equation in (1.5) should read: $\gamma=v/GD+\frac{1}{2}d(\ln GD)/d\theta$.]

J. P. LaSalle (Notre Dame, Ind.).

★ **Klotter, K. The attenuation of damped free vibrations and the derivation of the damping law from recorded data.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 85–93. American Society of Mechanical Engineers, New York, 1955. $9.00.

The primary problem is to determine an empirical damping term in a single degree of freedom system so as to fit a given experimental oscillation-decay curve. The Kryloff-Bogoliuboff method is to treat cases of weak damping not directly integrable. Practically speaking, the simplest damping terms possible are desired, so the author confines himself to polynomials in the absolute value of the dependent variable, of its derivative, or of its amplitude, multiplied by the sign of its derivative. Particular attention is given to single powers. Practical methods for fitting the curves to determine the empirical parameters are discussed. *E. Pinney* (Berkeley, Calif.).

Derwidué, L. Sur certaines équations de Hurwitz. Mathesis **65** (1956), 37–40.

Consider a system of differential equations of the form

$$A_n\frac{d^nx}{dt^n}+A_{n-1}\frac{d^{n-1}x}{dt^{n-1}}+\cdots+A_1\frac{dx}{dt}+A_0x=0.$$

A problem of importance is that of determining stability conditions without writing out the characteristic polynomial. The author gives two criteria, one for the case where the A_i commute, and one for the case where the A_i are hermitian. *R. Bellman* (Santa Monica, Calif.).

Mitrinovitch, D. S. Sur l'équation différentielle d'un problème de technique etudié par M. R. Gran Olsson. Norske Vid. Selsk. Forh., Trondheim **28** (1955), 171–175 (1956).

The author finds a general quadrature solution to the equation

$$f(x)(y^{(4)}-2a^2y''+a^4y)+2f'(x)(y'''-a^2y')=0 \quad (a=\text{const}),$$

$f(x)$ being an arbitrary function. He notes a large number of special cases to be found in Kamke's Differentialgleichungen. *E. Pinney* (Berkeley, Calif.).

Erugin, N. P. On the analytic theory of non-linear differential equations. Vestnik Leningrad. Univ. **11** (1956), no. 7, 60–70. (Russian)

The author considers the behavior of solutions of the special system $dx/dt=q(y)$, $dy/dt=p(x,t)$, as $t\to\infty$, under various assumptions concerning q and p. The results are closely connected with the Borel-Hardy-Fowler theory discussed in R. Bellman, "Stability theory of differential equations" [McGraw-Hill, New York, 1953; MR **15**, 794].

R. Bellman (Santa Monica, Calif.).

Caligo, Domenico. Sopra una classe di equazioni differenziali non lineari. Mem. Accad. Sci. Torino. Cl. Fis. Mat. Nat. (3) **1** (1955), 1–24.

Let $\Phi(y'', y', y)=0$ be a differential equation in which Φ is a homogeneous polynomial with constant coefficients in the quantities y'', y', y and the dash indicates differentiation with respect to the independent variable t. The author shows that such a differential equation can be integrated by means of quadratures. This is done by

introducing the functions $w=y''/y$, $v=y'/y$, which are linked by the relation (1) $\Phi(w, v, 1)=0$, and by the differential equation (2) $dv/dt=w-v^2$. If $\{w(s), v(s)\}$ is some parametric representation of (1) the equation (2) permits the calculation of $s(t)$, and then $y(t)$ is found from $v=y'/y$. An analogous technique can be applied to higher order differential equations of similar structure. The main part of the paper is devoted to an exhaustive analysis of the case that Φ is quadratic. A detailed study of the singularities of the solutions and several examples are included. *W. Wasow* (Los Angeles, Calif.).

Reissig, Rolf. Über die Stabilität gedämpfter erzwungener Bewegungen mit linearer Rückstellkraft. Math. Nachr. 13 (1955), 231–245.

The problem considered, the results obtained, and the methods used, are similar to those of a previous series of papers by the author [Math. Nachr. 11 (1954), 231–238, 345–384; 12 (1954), 283–300, 119–128; MR 16, 250, 1112]. *J. P. LaSalle* (Notre Dame, Ind.).

Reissig, Rolf. Über die Stabilität gedämpfter erzwungener Bewegungen mit linearer Rückstellkraft (Ergänzung). Math. Nachr. 14 (1955), 17–20.

The author continues his study of a second-order differential equation related to the forced oscillation of a system of one degree of freedom with nonlinear viscous damping, Coulomb damping, and an elastic restoring force. In a previous paper reviewed above sufficient conditions were given for the existence of a unique periodic solution to which all other solutions are asymptotic as $t\to+\infty$. In this paper the conditions on the nonlinear viscous damping are weakened slightly but not significantly. Essentially the condition on the damping is the requirement that it be a positive resistance.

J. P. LaSalle (Notre Dame, Ind.).

Tuzov, A. P. On the stability of certain periodic motions. Leningrad. Gos. Univ. Uč. Zap. **144.** Ser. Mat. Nauk 23 (1952), 247–256. (Russian)

The author discusses Lyapunov's result on the stability of periodic motions in the neighborhood of a singular point when two of the characteristic roots are pure imaginary and all the others have negative real parts. The result states that under certain conditions the non-perturbed motion (the singular point) is stable and all motions sufficiently near the non-perturbed motion are asymptotic to periodic motions. The author points out that Lyapunov's conditions are unnecesarily restrictive and proves the result with one of the restrictions omitted.

J. P. LaSalle (Notre Dame, Ind.).

Hohlov, R. V. On the synchronization of the external force for two connected self-oscillating systems. Vestnik Moskov. Univ. 11 (1956), no. 3, 41–49. (Russian)

The systems under consideration are of the type

$$\ddot{x}-2\delta_0(x)\dot{x}+\nu_1^2x+\alpha_1\ddot{y}=0,$$

$$\ddot{y}-2\gamma_0(y)\dot{y}+\nu_2^2y+\alpha_2\ddot{x}=E\nu_2^2\cos\omega t.$$

Approximate solutions are

$$x=A\cos(\omega t-\psi),\quad y=B\cos(\omega t-\varphi),$$

where A, B, φ, ψ vary slowly with the time and satisfy

$$\dot{A}=\delta(A)A-\tfrac{1}{2}\alpha_1\omega B\sin(\varphi-\psi),$$

$$\dot{B}=\gamma(B)B+\tfrac{1}{2}\alpha_2\omega A\sin(\varphi-\psi),$$

$$\dot{\psi}=\Delta_1+\tfrac{1}{2}\alpha_1\omega\frac{B}{A}\cos(\varphi-\psi),$$

$$\dot{\varphi}=\Delta_2+\tfrac{1}{2}\alpha_2\omega\frac{A}{B}\cos(\varphi-\psi)+\frac{E\omega}{2B}\cos\varphi,$$

$$\Delta_i=\omega-\nu_i.$$

It is assumed furthermore that α_1, α_2, $E\omega$ are small relative to δ/ω, γ/ω, γB_0. Introducing a small multiplier μ the first approximation is taken and readily solved and discussed. Noteworthy case: E is of order μ^2.

S. Lefschetz (Mexico, D.F.).

Minorsky, Nicolas. Sur l'interaction des oscillations non-linéaires. Rend. Sem. Mat. Fis. Milano **25** (1953–54), 145–163 (1955).

The differential equation

$$\ddot{x}+\epsilon(x^2-1)\dot{x}+[1+(a+cx^2)\cos 2t]x=0 \tag{1}$$

in which ϵ, a and c are parameters independent of t reduces to Van der Pol's equation when $a=c=0$, and to an equation called non-linear Mathieu equation by the author when $\epsilon=0$. The author gives a detailed approximate discussion of the stable periodic solutions of this equation for small parameter values emphasizing in particular the relation of these solutions to those of the two simpler equations mentioned above. The study is based on the author's so-called stroboscopic method [Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **9** (1952), 23–29; MR **16,** 477] which reduces the approximate determination of the periodic solutions of (1) to that of the singular points of a certain simpler auxiliary differential system. The study of this auxiliary system shows that there may exist as many as three periodic solutions. However, the existence and stability of these solutions depends on certain inequalities between the parameters, which are discussed in great detail. If more than one stable periodic solution is possible, and if the system starts from its rest position, a topological analysis in the phase plane shows which oscillation will actually take place. As the parameters vary, discontinuous transitions from one oscillatory state to another may occur. For certain parameter ranges the energy of the oscillation is the same as for the corresponding Van der Pol equation; for certain other ranges it coincides with the energy corresponding to Mathieu's equation. The author describes this situation by saying that one oscillator "suffocates" the other. *W. Wasow.*

Reissig, Rolf. Über eine nichtlineare Differentialgleichung 2. Ordnung. III. Math. Nachr. **15** (1956), 39–45.

[For parts I–II see Math. Nachr. 13 (1955), 313–318; 14 (1955), 65–71; MR 17, 38, 739.]

The differential equation is of the form

$$\ddot{u}(t)+F(\dot{u}(t))+G(u(t))=E(t),$$

where $E(t)$ is periodic. Sufficient conditions for the existence of a periodic solution with period that of E are given. Existence is established by use of Brouwer's fixed-point theorem. *J. P. LaSalle* (Notre Dame, Ind.).

Reissig, Rolf. Über eine nichtlineare Differentialgleichung 2. Ordnung. IV. Math. Nachr. **15** (1956), 47–54.

The study is similar to that reviewed above with a slightly different set of initial conditions. *J. P. LaSalle.*

Berezanskiĭ, Yu. M. On the inverse problem of spectral analysis for the Schrödinger equation. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 197–200. (Russian)

The inverse problem is considered for $-\Delta u+c(p)u=\lambda u$ in a region G with boundary condition $\partial u/\partial n+\sigma(p)u=0$, where n is the normal to the boundary surface of G and $\sigma(p)$ is given on the boundary. A function $f(p)$ is $f(x, y, z)$. The results are given in terms of the spectral function $\theta(p, q; \lambda)$ where p and q are points of G and $-\infty<\lambda<\infty$. The relationship between $c(p)$ and $\theta(p, q; \lambda)$ is considered as is the determination of $c(p)$ for $\theta(p, q; \lambda)$.

N. Levinson (Cambridge, Mass.).

Kreĭn, M. G. On determination of the potential of a particle from its S-function. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 433–436. (Russian)

Let $S(k)$, $-\infty<k<\infty$, satisfy $|S(k)|=S(0)=1$, $S(-k)=\bar{S}(k)$, $\arg S(k)\to\arg S(0)$ and $k\to\infty$, and

$$S(k)=1+\int_{-\infty}^{\infty} e^{ikt}s(t)\,dt,$$

where $s(t)\in L(0, \infty)$. Then $S(k)$ is shown to be the S-function associated with a one dimensional wave equation. Other results are given. *N. Levinson.*

Hartman, Philip; and Putnam, Calvin R. The essential spectrum and averages of the potential. Duke Math. J. **23** (1956), 83–91.

Let $q(t)$ be real and continuous. The spectrum $S(\alpha)$ of $x''+(\lambda+q(t))x=0$, $0\leq t<\infty$, with boundary condition $x(0)\cos\alpha-x'(0)\sin\alpha=0$ is considered in the limit point case. The set of cluster points of $S(\alpha)$ is denoted by S' and is known to be independent of α. The authors discuss the relationship of S' and the behavior as $t\to\infty$ of $Q(t)=M-\int_0^t q(s)ds$, where M is a constant. They show that if for some M as $t\to\infty$ $\liminf T^{-1}\int_0^T Q^2(t)dt=0$, then S' contains $[0, \infty)$. It is also shown that the differential equation is of limit point type if as $t\to\infty$, $\liminf T^{-3}\int_0^T Q^2(t)dt<\infty$. Many other results involving $Q(t)$ are given. Weighted means of $q(t)$ involving $y(t)$, a solution of $y''+q(t)y=0$, also enter as criteria for theorems. *N. Levinson.*

Looman, H. Observations about some differential equations concerning recession of mountain slopes. I, II. Nederl. Akad. Wetensch. Proc. Ser. B. **59** (1956), 259–271, 272–284.

Pliś, A. On the estimation of the existence domain for solutions of a non-linear partial differential equation of the first order. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 125–129.

The initial value problem for a non-linear partial differential equation of first order

$$\frac{\partial z}{\partial x}=f\left(x, y_1, \cdots, y_n, z\frac{\partial z}{\partial y_1}, \cdots, \frac{\partial z}{\partial y_n}\right)$$

can be solved, according to Cauchy, on any (one-sided) neighborhood of the initial manifold M which is simply covered by the characteristic curves (*) $y_i=y_i(x, V)$ issuing from the points $V=(v_1, \cdots, v_n)$ of M. The author's aim is to estimate the possible size of such a neighborhood, on which relations (*) can be inverted. For this purpose, he considers an augmented system of characteristic equations which include, besides the usual ones concerning presumptive values of z and $\partial z/\partial y_i$ on the characteristic curves, the analogous equations relating to the second derivatives $\partial^2 z/\partial y_i\partial y_j$, and he justifies the following criterion: Where this augmented system of characteristic equations has a solution satisfying certain suitable initial conditions, the functional determinant $|\partial y_i/\partial v_j|$ will not be zero. This criterion is somewhat simplified in the case in which f is independent of z.

A. Douglis (New York, N.Y.).

Ważewski, T. Sur la méthode de A. Pliś de déterminer le domaine d'existence de la solution du problème de Cauchy pour les équations aux dérivées partielles du premier ordre. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 131–135.

In this paper, the results of the paper reviewed above are clearly formulated and elegantly proved.

A. Douglis (New York, N.Y.).

Nicolesco, Miron. Sur un théorème de M. J. Hadamard. C. R. Acad. Sci. Paris **243** (1956), 13–15.

Let D be a domain the (x, t)-plane bounded by the lines $t=t_j$, $j=1, 2$, $t_1<t_2$, and the two curves $x=\varphi_j(t)$, $(\varphi_1(t)<\varphi_2(t)$ for $t_1<t<t_2)$. Suppose the continuous functions $\varphi_j(t)$ are such that the first boundary value problem for the heat equation (*) $u_{xx}-u_y=0$ is always solvable with assigned continuous boundary values on that portion of the boundary of D which lies below the line $t=t_2$. A function which in D satisfies a certain differential equation and which is continuous together with its derivatives which appear in that differential equation will be called a regular solution in D of the equation. Let $\{u_n(x, y)\}$ be a family of regular solutions of equation (*) in D such that (a) $u_n>0$ for each n and (b) $\sum_1^\infty u_n$ converges at the boundary point (x, t_2) of D then it has been shown that $\sum_1^\infty u_n$ converges uniformly in each domain $D'\in D$ [J. Hadamard, Rend. Circ. Mat. Palermo (2) **3** (1954), 337–346; MR **16**, 930; B. Pini, Rend. Sem. Mat. Univ. Padova **23** (1954), 422–434; MR **16**, 485]. The author states that he has extended this result to solutions of the equation (**) $L^p(u)\equiv(\partial^2/\partial x^2-\partial/\partial t)^p u=0$. In more detail: If $\{u_n\}$ is a family of regular solutions of (**) in D, and if (a) $(-1)^k L^k(u_n)>0$, $k=0, 1, \cdots, p-1$, $n=1, 2, \cdots$, and (b) the $\sum_1^\infty u_n$ converges at the boundary point (x, t_2) of D, then this series converges uniformly in each domain D' contained in D. *F. G. Dressel.*

Consiglio, Alfonso. Sulla canonicità di un sistema di equazioni differenziali di tipo normale. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 153–160.

It is observed that necessary and sufficient conditions for the canonicity of a system of $2n$ differential equations of the first order can be written in the form of $\frac{1}{2}n(n-1)$ conditions of integrability for the Hamiltonian. Known results on linear systems are obtained by particularization.

D. C. Lewis (Pacific Palisades, Calif.).

See also: Erdélyi, p. 1202; Perls and Sherrard, p. 1242; Geis, p. 1248; Weidenhammer, p. 1254.

Partial Differential Equations

★ **Duff, G. F. D. Partial differential equations.** Mathematical expositions no. 9. University of Toronto Press, Toronto, 1956. x+248 pp. $6.50.

Ce livre est consacré aux équations aux dérivées partielles du premier et du second ordre. Certaines questions

sont traitées dans tous les détails, plusieurs sont étudiées plus sommairement, d'autres enfin ne sont pas abordées. Ainsi, la théorie des équations aux dérivées partielles du premier ordre et son application à la théorie des caractéristiques pour les équations hyperboliques normales font l'objet de développements circonstanciés. Il en est de même des équations elliptiques du second ordre (où les questions d'existence sont traitées via la théorie des équations intégrales). On s'attendrait, en contre-partie, à voir développer, avec la même ampleur, les problèmes aux limites pour les équations hyperboliques et paraboliques du second ordre. Cependant, il n'est question de ces problèmes que dans un paragraphe et seul, le cas particulier du problème de Cauchy pour l'équation des ondes fait l'objet d'un chapitre entier. Ajoutons que cette dernière question est étudiée par la méthode du prolongement analytique, dont l'auteur semble reconnaître le caractère artificiel (alors que des rudiments de théorie des distributions de L. Schwartz permettraient de résoudre ce problème d'une manière élémentaire et naturelle). D'un autre côté, le chapitre relatif aux fonctions et valeurs propres des opérateurs elliptiques du second ordre paraît un peu sacrifié dans un ouvrage notamment destiné à ceux qui s'intéressent aux applications.

Il serait injuste cependant de ne pas reconnaître les nombreuses qualités de l'exposé. Facilement accessible, il constitue une introduction utile aux parties classiques de la théorie des équations aux dérivées partielles. L'impression et la présentation sont excellentes. L'ouvrage contient des exercices et des indications bibliographiques.

Table des matières: I. Differential equations and their solutions, II. Linear equations of the first order, III. Non-linear equations of the first order, IV. Linear equations of the second order, V. Self-Adjoint elliptic equations, VI. Linear integral equations, VII. Boundary value problems, VIII. Eigenfunctions, IX. Normal hyperbolic equations, X. Integration of the wave equation. *H. G. Garnir.*

★**Lense, J. Partielle Differentialgleichungen. Handbuch der Physik. Bd. I.** Mathematische Methoden I, pp. 90–119. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. DM 72.00.

This is a survey of the general theory of partial differential equations. Topics treated include, among others, linear and quasi-linear equations of the first order, characteristics and characteristics strips, complete integrals, contact transformations, the Hamilton-Jacobi equation, canonical equations and transformations, total differential equations, Pfaffian forms, integral invariants, general and semi-linear equations of the second order with two independent variables, and the Riemann function for the hyperbolic equation. *F. A. Ficken.*

Zaidman, S. Théorèmes qualitatifs pour des équations aux dérivées partielles. Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 645–666. (Romanian. Russian and French summaries)

Applying the methods of semi-groups of operators, the author obtains a series of qualitative theorems on the stability and asymptotic almost periodicity of the solutions to a parabolic system of equations, the wave equation, and the Schrödinger equation. The parabolic case is typical. Here he considers the linear system $\partial u/\partial t = Au$ where A is a strongly elliptic operator on vector functions $u(x)$ defined on a bounded domain Π in n-dimensional euclidean space. The operator A is defined on a subspace $D(A)$ of the Hilbert space $H(\Pi)$ with vanishing boundary conditions. V. È. Lyance [Mat. Sb. **35(77)** (1954), 357–368; MR **16**, 709] has shown that $\mathrm{Re}\,(Au, u) \leqq \beta(u, u)$ for all $u \in D(A)$ and $\mathrm{Re}\,(A^*v, v) \leqq \beta(v, v)$ for all $v \in D(A^*)$ and this leads immediately to the fact that A generates a semi-group of operators $[T(t)]$ with $\|T(t)\| \leqq \exp(\beta t)$. Suppose that $F(x, t, u)$ satisfies a Lipschitz condition $\|F(\cdot, t, u_1) - F(\cdot, t, u_2)\| \leqq g(t)\|u_1 - u_2\|$ for $u_1, u_2 \in D(A)$. Then it is shown that the solution of $\partial u/\partial t = Au + F(\cdot, t, u)$ is stable for $\beta = 0$ and $\int_0^\infty g(t)dt < \infty$ and asymptotically stable if $g(t) = L$ and $L + \beta < 0$. Another theorem asserts that the solution of $\partial u/\partial t = Au$ is asymptotically almost periodic for $\beta \leqq 0$; here the proof seemed obscure to the reviewer. The wave and Schrödinger equations are treated similarly. *R. S. Phillips.*

Morgenstern, Dietrich. Singuläre Störungstheorie partieller Differentialgleichungen. J. Rational Mech. Anal. **5** (1956), 203–216.

Let H be a linear Hilbert space and H_1 a subspace. Let D and D_1 be submanifolds of H and H_1 respectively. Let B be a positive definite quadratic form in H and A in H_1. It is further assumed that D_1 is dense in D with respect to B. The variational problem is to minimize $\varepsilon A(\phi) + B(\phi - \chi)$ for ϕ in D_1, $\varepsilon > 0$ and χ a fixed element of H. The related problem to minimize $B(\phi - \chi)$ for ϕ in D is also considered. The solutions are denoted by u_ε and v respectively. Here the relationship between u_ε and v as $\varepsilon \to 0$ are considered. Applications are made to partial differential equations with boundary conditions. The equations $-\varepsilon\Delta u + u = b(x)$ and $\varepsilon\Delta(p(x)\Delta u) - \Delta u + a(x)u = b(x)$ are considered as applications of the general theory. *N. Levinson* (Cambridge, Mass.).

Ghermănescu, M. Sur l'intégration d'un système d'équations aux dérivées partielles. Com. Acad. R. P. Romîne **5** (1955), 1703–1706. (Romanian. Russian and French summaries)

Elementary considerations of a system of linear partial differential equations arising from a paper of Vrănceanu [Com. Acad. R. P. Romîne **4** (1954), 335–340; MR **17**, 406]. *R. Blum* (Saskatoon, Sask.).

Rusak, B. I. On the construction of a new integral of the Lagrange equations of second order with two data. Dokl. Akad. Nauk Uzbek. SSR. **1953**, no. 6, 7–11. (Russian. Uzbek summary)

Sind $\varphi(q_k, \dot{p}_k) = \text{const}$ und $\psi(q_k; \dot{p}_k) = \text{const}$ zwei Integrale eines Hamiltonschen Gleichungssystems, so ist mit der Poissonschen Klammerbildung (φ, ψ) durch $(\varphi, \psi) = \text{const}$ ein drittes Integral des Systems gegeben. Verfasser beweist den Satz: sind $\Phi(q_k; \dot{p}_k) = \text{const}$ und $\Psi(q_k; \dot{p}_k) = \text{const}$ zwei Integrale des Lagrangeschen Gleichungssystems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, 2, \cdots, n,$$

so ist auch

$$\frac{1}{D}\sum_k\sum_m (-1)^{k+m} D_{km}\left(\frac{\partial \varphi}{\partial q_k}\frac{\partial \Psi}{\partial \dot{q}_m} - \frac{\partial \Psi}{\partial q_k}\frac{\partial \Phi}{\partial \dot{q}_m}\right) + \frac{1}{D^2}\sum_k\sum_m\sum_n\sum_p (-1)^{k+m+n+p} D_{km} D_{np} \times \frac{\partial^2 L}{\partial q_k \partial \dot{q}_n}\left(\frac{\partial \Phi}{\partial \dot{q}_m}\frac{\partial \Psi}{\partial \dot{q}_p} - \frac{\partial \Phi}{\partial \dot{q}_p}\frac{\partial \Psi}{\partial \dot{q}_m}\right) = \text{const}$$

ein Integral dieses Systems. Dabei bedeutet D die De-

terminante der Elemente $a_{km}=\partial^2 L/\partial\dot{q}_k\partial\dot{q}_m$ und D_{km} die zugehörigen Minoren. Als Spezialfall werden die Ableitungen $\partial^2 L/\partial q_k\partial\dot{q}_m=L_{k\dot{m}}$ symmetrisch angenommen ($L_{k\dot{m}}=L_{m\dot{k}}$). Mit $L=\sum_m a_m\dot{q}_m^2+f(q_k)$ und $a_m=$const kommt man wieder auf Poissonsche Klammerbildungen zurück. *M. Pinl* (Köln).

Šul'gin, M. F. Generalization of Poisson's theorem to the case of an arbitrary system of differential equations. Dokl. Akad. Nauk Uzbek. SSR. **1953**, no. 11, 3–7. (Russian. Uzbek summary)

Setzt man nach Liouville

$$\mathfrak{H}=\sum_{i=1}^{n} f_i(t; x_1, x_2, \cdots, x_n)y_i,$$

so erhält man anstelle des Systems

$$(*) \qquad \dot{x}_k=f_k(t; x_1, x_2, \cdots, x_n),\ k=1, 2, \cdots, n,$$

das System

$$(**) \qquad \dot{x}_k=\frac{\partial\mathfrak{H}}{\partial y_k},\ k=1, 2, \cdots, n,$$

und für die überzähligen Veränderlichen $y_1, y_2, \cdots, y_n$ das System

$$(***) \qquad \dot{y}_k=-\frac{\partial\mathfrak{H}}{\partial x_k}=-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_k}y_i,\ k=1, 2, \cdots, n.$$

Die beiden Systeme (**), (***) bilden ein Hamiltonsches System von $2n$ Gleichungen mit der charakteristischen Funktion

$$\mathfrak{H}(t; x_1, \cdots, x_n; y_1, \cdots, y_n).$$

Jedes Integral $\varphi(t; x_k)=$const des Systems (*) genügt der Identität

$$\frac{\partial\varphi}{\partial t}+\sum_{k=1}^{n}\frac{\partial\varphi}{\partial x_k}f_k\equiv 0$$

und die Ableitungen $\partial\varphi/\partial x_k$ sind Lösungen y_k des Systems (***). Nunmehr geht Verfasser von zwei Integralen $\varphi(t; x_i)=$const und $\psi(t; x_i, y_i)=$const, $i=1, 2, \cdots, n$, des Systems (**), (***) aus. Dann ist nach einem klassischen Satz von Poisson auch der Klammerausdruck

$$(\varphi, \psi)=\sum_{i=1}^{n}\frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial y_i}=\text{const}$$

ein Integral dieses Systems. Im allgemeinen wird (φ, ψ) von den Veränderlichen x_i, y_i, t abhängen. Da jedoch $\varphi=$const ein Integral von (*) und $\partial\varphi/\partial x_k=y_k$ eine Lösung des Systems (***) darstellen, können die $y_1, y_2, \cdots, y_n$ eliminiert und ein neues Integral des Systems (*) gewonnen werden. Durch diese Verallgemeinerung des Poissonschen Satzes erhält Verfasser von einem Integral des Systems (**) (***) und einem solchen des Systems (*) ein zweites Integral des Systems (*), damit weiter ein drittes usw. In günstigen Fällen gelingt es auf diese Weise ein vollständiges Integralsystem der Gleichungen (*) zu finden. Als wichtige Spezialfälle werden weiterhin skleronome Systeme behandelt, bei welchen keine der Funktionen f_k von t explizit abhängt. Ist dann $\varphi(t; x_i)=$const ein Integral des skleronomen Systems, so auch

$$\frac{\partial\varphi}{\partial t}=c,\ \frac{\partial^2\varphi}{\partial t^2}=c_1,\ \frac{\partial^3\varphi}{\partial t^3}=c_2,\ \text{usw.}$$

Weitere interessante Fälle ergeben sich, wenn sogenannte zyklische Variable auftreten. Als Musterbeispiel wird zum Schluss das System

$$\dot{x}_1=-x_2+x_3,\ \dot{x}_2=x_3,\ \dot{x}_3=-x_1+x_3$$

behandelt. *M. Pinl* (Köln).

Landis, E. M. On some properties of solutions of elliptic equations. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 640–643. (Russian)

The author considers the second order linear elliptic equation

$$\sum_{i,k=1}^{n} a_{ik}\frac{\partial^2 u}{\partial x_i\partial x_k}+\sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i}+cu=0$$

with variable coefficients a_{ik} and b_i, assumed C'' and C', respectively, and bounded on a region G. It is further assumed that on this region

$$\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k>\alpha>0 \text{ for } \sum_{i=1}^{n}\xi_i^2=1.$$

Uniqueness of the solution of the Cauchy problem, continuous dependence of the solution of the Cauchy problem on the initial conditions, and certain growth properties of the solution are arrived at by means of the inequality contained in the following theorem. Let S be the unit sphere in n-space, G the interior region bounded by S, and γ the portion of S determined by

$$\sum_{i=1}^{n} x_i^2=1,\ \sum_{i=2}^{n} x_i^2\leqq a^2<\tfrac{1}{4},\ x_1>0.$$

Then there exists a constant C depending only on α and the dimensionality n, such that for any $\varepsilon>0$ and any solution u continuous on $\bar{G}$ and having a normal derivative on γ the conditions

$$|u|\leqq 1 \text{ on } S,\ |u|<\varepsilon \text{ on } \gamma,\ |\partial u/\partial n|<\varepsilon \text{ on } \gamma$$

imply

$$|u(0)|<\varepsilon^{a^C}.$$

M. G. Arsove (Seattle, Wash.).

Lax, Peter D. On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations. Comm. Pure Appl. Math. **8** (1955), 615–633.

Let $A=\sum A_kD_k+A_0$ ($D_k=\partial/\partial x_k$) be a real linear differential operator of order one whose coefficients are square matrices depending on x. Assume that A is symmetric hyperbolic with respect to the first variable x_1, i.e. that $\{A_k\}_1^n$ are symmetric and A_1 positive definite. The author solves Cauchy's problem for A with initial data on a plane $x_1=$const without using Cauchy-Kowalevski's theorem in a manner different from that of Friedrichs [same Comm. **7** (1954), 345–392; MR **16**, 44]. The compactness of the domain of dependence makes it possible to reduce the problem to solving $Au=v$ with A, u and v periodic in all variables and $A_0+A_0^*$ sufficiently positive (attained initially by a transformation $u\to e^{tx_1}u$, which transforms A_0 to A_0+tA_1). Consider the scalar product $(D^su, D^sv)=\sum(D^\alpha u, D^\alpha v)$, where D^α runs over the set D^s of all derivatives of order $|\alpha|\leqq s$ and (u, v) is the ordinary scalar product in L^2. C^∞ is dense in the corresponding Hilbert space H^s. Put $|D^su|=(D^su, D^su)^{\frac{1}{2}}$ and $\Delta_s=\sum(-1)^{|\alpha|}D^\alpha D^\alpha$ ($|\alpha|\leqq s$). That $A_0+A_0^*$ is sufficiently positive means that for arbitrarily large s we have (!) $(A\Delta_su, u)\sim(D^sAu, D^su)\geqq|D^su|^2$ provided the coefficients of A are in C^s. Now let H^{-s} be the dual space of H^s with respect to (an extension of) the duality (u, v). It is a space of distributions and the corresponding norm is $|D^{-s}u|=\sup(u, v)/|D^sv|$. It is clear that C^∞ is dense in H^{-s}, that $\Delta_sC^\infty=C^\infty$ and that $|D^{-s}\Delta_su|=|D^su|$ so that Δ_s can be extended to a linear homeomorphism $H^s\leftrightarrow H^{-s}$. (There is no reference to the fact that the spaces H^{-s} have been

introduced by L. Schwartz). The fundamental result is now that (2) $|D^s u| \leq \text{const}\,|D^s A u|$ for s positive, null and negative. For $s \geq 0$ this follows from (1), and combining the first and third terms of (1) we get $|D^s u| \leq \text{const}\,|D^{-s} A \Delta_s u|$ which implies (2) with s changed to $-s$ if we replace u by $\Delta_s^{-1} u$. Corollary: $Au = v \in H^s$, $(s>0)$, has a unique solution u in H^s. Proof: apply (2) to $A^* = -\sum D_k A_k + A_0^*$. Let $f \in C^\infty$. Because $|(v, f)| \leq |D^s v|\,|D^{-s} f| \leq \text{const}\,|D^s v|\,|D^{-s} A^* f|$, (v, f) is a bounded linear function of $A^* f \in H^{-s}$. Extending it boundedly to all of H^{-s} (actually, $A^* C^\infty$ is dense in H^{-s}) we see that $u \in H^s$ exists such that $(v, f) = (u, A^* f)$; u is the required solution. — In the last part of the paper, the interior differentiability of weak solutions of elliptic operators is proved with the same technique. (Reviewer's remark. It is not difficult to get, in the same fashion, the differentiability on the boundary). *L. Gårding* (Lund).

Dezin, A. A. Mixed problems for certain symmetric hyperbolic systems. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 13–16. (Russian)

Let the differential operator $A = \sum_1^n A_k D_k + A_0$ be symmetric hyperbolic [see the preceding review] and let $P \neq 0, 1$ be a constant projection such that $0 = PA_kP = (1-P)A_k(1-P)$ when $k>1$. Let C be a cube with sides parallel to the coordinate planes; label them $C_k^{\pm}$ $(k=1, \cdots, n)$. Consider the boundary value problem $Au = f$ with $u=0$ on C_1^- and $PA_ku=0$ (or $(1-P)A_ku=0$) on $C_k^{\pm}$ $(k>1)$. It is shown by a direct argument similar to that of Friedrichs [Comm. Pure Appl. Math. **7** (1954), 345–392; MR **16**, 44] that the problem has a unique solution. Special cases are Maxwell's equations and the linearized hydrodynamical system. *L. Gårding.*

Duff, G. F. D. Modified boundary value problems for a quasi-linear elliptic equation. Canad. J. Math. **8** (1956), 203–219.

Let D be a domain in N space, B its boundary and let P denote points in $D+B$. The quasi-linear elliptic differential equation (*) $\Delta u = -F(P, u)$ is discussed where F is continuous in P and u, positive and increasing with u. It is shown that the Dirichlet problem has not always a solution; but if a continuous positive function f is prescribed on B and a positive number M assigned, there exists always a solution of (*) which has precisely the maximum M and is on B proportional to f. The proof is based on the Schauder-Leray theorem applied to a nonlinear integral equation with an auxiliary parameter. Similarly, it is shown that for prescribed $M>0$ and a positive function $f \leq M$ on B there exists a factor t such that $\Delta u = -tF(P, u)$ has a solution with maximum M and boundary values $f(P)$. Further extensions and treatment of Neumann type boundary value problems are given. In each case a parameter t is admitted but the maximum M for the solution prescribed. *M. Schiffer.*

Miranda, Carlo. Sul problema misto per le equazioni lineari ellittiche. Ann. Mat. Pura Appl. (4) **39** (1955), 279–303.

Let R_m be the m-dimensional real Euclidean space with points $x = (x_1, \cdots, x_m)$. Let Σ be a bounded set in the plane $x_m = 0$ and V its boundary. Let T be a domain in R_m which is situated in $x_m \geq 0$ and whose boundary is the union of Σ and a set S whose intersection with $x_m = 0$ is V. The author considers the following problem: To find a solution $u = u(x)$ in T of the elliptic equation

$$\sum_{i,k=1}^{m} a_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f(x$$

which satisfies the mixed boundary condition: On S, u is given while on Σ the expression

$$P(u) \equiv -\sum_{k=1}^{m} a_{mk} \frac{\partial u}{\partial x_k} + \beta(x)u$$

is given. Under very general conditions on the given data, it is proved that the theorem of the alternative holds for this problem and that particularly in the case $c \leq 0$, $\beta \geq 0$ there exists one and only one solution with certain differentiability and Hölder properties. Concerning a comparison of the results and hypotheses of the author with those of previous authors on the mixed boundary value problem we refer to the introduction of the paper. The method used is the now classical one of establishing a priori estimates first for the Poisson equation, then for equations with constant coefficients and finally for the given equation, and accordingly the main body of the paper deals with such delicate estimates. *E. H. Rothe.*

Pini, Bruno. Osservazioni sopra un problema generalizzato di Dirichlet per le equazioni lineari del secondo ordine ellittiche e paraboliche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 237–246.

Let D denote a domain in the xy-plane bounded by the smooth curve C: $x = x(s)$, $y = y(s)$, $0 \leq s \leq 1$. For each fixed value of t $(0 < t \leq h)$ denote by C_t the curve interior to D with equation $x = x(s) - ty'(s)$, $y = y(s) + tx'(s)$, $0 \leq s \leq 1$. G. Cimmino [Rend. Circ. Mat. Palermo **61** (1937), 177–221, prove the existence of a solution of the following generalized Dirichlet problem: Given $p > 1$ and a function $f(s)$ of class $L^{(p)}$ on $0 \leq s \leq 1$, determine a function $u = u(x, y)$ such that

$$(*) \qquad \Delta u = 0, \text{ in } D, \lim_{t \to 0+} \int_{C_t} |u - f(s)|^p ds_t = 0.$$

E. Magenes [Rend. Sem. Mat. Univ. Padova **24** (1955), 220–229; MR **17**, 28] showed that the solution of the above problem could be represented as the potential of a double layer. The author proves that these results of Cimmino and Magenes also hold for the case $p=1$. He then shows that similar theorems hold for $p \geq 1$ when Δ in (*) is replaced by the elliptic operator

$$L = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + m\frac{\partial}{\partial x} + n\frac{\partial}{\partial y} + q,$$

where $a>0$, $ac - b^2 = 1$, $q<0$, $a_{xx} + 2b_{xy} + c_{yy} - m_x - n_y + q < 0$, and the coefficients of operator L and its adjoint are of class C''.

In the last part of this paper corresponding results to those mentioned for the elliptic equation are shown to hold for a similar generalization of the first boundary value problem of the parabolic equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} + a\frac{\partial u}{\partial x} + bu = 0, \quad b<0, \quad b - a_x < 0.$$

[See also, B. Pini Ann. Mat. Pura Appl. (4) **32** (1951), 179–204; MR **13**, 750.] *F. G. Dressel* (Durham, N.C.).

Koševlev, A. I. On boundedness in L_p of derivatives of solutions of elliptic differential equations. Mat. Sb. N.S. **38(80)** (1956), 359–372. (Russian)

Let L be a real elliptic differential operator of the second order defined in a bounded region Ω in n-space. Put $|D^k f|_p = \sum (\int |Df|^p d\Omega)^{1/p}$, where the sum extends over all the derivatives Df of order $\leq k$. Ladyženskaya [Dokl.

Akad. Nauk SSSR (N.S.) **79** (1951), 723–725; MR **14**, 280] has shown that if u vanishes at the boundary Γ of Ω then $|D^2u|_2 \leqq |Lu|_2$, (c a constant), provided that Γ and the coefficients of L are smooth enough and that the equation $Lu=0$, $u=0$ on Γ, has no solution $\neq 0$. In this paper it is shown that $|D^2u|_p \leqq c|Lu|_p + c \sup |u, T|_p$ ($p>1$), where T is a plane section of Ω and $|u, T|_p$ the corresponding p-norm obtained by integrating over T. If $p>2(1-n^{-1})$ and L is self-adjoint, the last term can be cancelled.

L. Gårding (Lund).

Corduneanu, C. La dépendance des solutions des équations hyperboliques par rapport aux coefficients et aux données sur les caractéristiques. Acad. Repub. Pop. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 313–317. (Romanian. Russian and French summaries)

Après avoir donné un exemple où la solution du problème:

$$z_{xy}+a(x, y)z_x+b(x, y)z_y+c(x, y)z=d(x, y),$$
$$z(x, 0)=f(x),\ z(0, y)=g(y)$$

ne dépend pas continuement des données sur les caractéristiques $f(x)$ et $g(y)$, l'auteur montre qu'à de petits accroissements (dans le sens de la métrique de Tchebychev) de a, b, c, d, f et g correspondent de petits accroissements de $z(x, y)$ si les variations totales de $f(x)$ et $g(y)$, ou de a par rapport à x et de b par rapport à y, subissent aussi de petits accroissements. *Y. Fourès-Bruhat.*

Ladyženskaya, O. A. The first boundary problem for quasilinear parabolic equations. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 636–639. (Russian)

Soit Ω un ouvert borné de R^n ($x=(x_1, \cdots, x_n) \in \Omega$) convenablement régulier. On cherche $u(x, t)$, $x \in \Omega$, $t \in [0, T]$, solution (sous des hypothèses de régularité convenables) de

$$(*)\quad \frac{\partial u}{\partial t}-\sum a_{ij}(x, t, u)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum a_i(x, t, u)\frac{\partial u}{\partial x_i}+a(x, t, u)=0,$$

avec $u(x, 0)=\varphi(x)$ donnée, $u(x, t)=0$ si $x \in$ frontière de Ω, où $\sum a_{ij}(x, t, u)\xi_i\xi_j \geqq \alpha \sum \xi_i^2$, $\alpha>0$, a_{ij}, a_i et a étant des fonctions convenablement régulières des „trois" variables x, t, u. L'A. montre que ce problème admet une solution unique, et étudie la régularité de la solution. Les démonstrations reposent sur une idée de E. Rothe [Math. Ann. **102** (1930), 650–670]. En bref on remplace $\partial u/\partial t$ par $h^{-1}(u(x, ph)-u(x, (p-1)h))$; dans (*) on remplace $u(x, t)$ par $u(x, (p-1)h)$ dans les coefficients a_{ij}, a_i et a et les termes $\partial^2 u/\partial x_i \partial x_j$ etc... par $\partial^2 u(x, ph)/\partial x_i \partial x_j$ etc...; d'où des équations linéaires en $u(x, ph)$; il reste alors à établir des majorations permettant de passer à la limite, ce que l'A. indique brièvement. *J. L. Lions* (Nancy).

Boigelot, A. M.; et Garnir, H. G. Sur les solutions de l'équation de la diffusion. Bull. Soc. Roy. Sci. Liège **25** (1956), 50–61.

Soit Ω un ouvert quelconque de R^n; soit u une distribution dans le cylindre $\Omega \times R_t$, nulle pour $t<0$, solution de $(-\Delta+D_t)u=m+u_0 \otimes \partial$, où m est une distribution nulle pour $t<0$ et ∂ la masse de Dirac à l'origine sur R_t. On suppose également que m est localement sommable dans $\Omega \times (t \geqq 0)$. Les A. montrent alors ceci: si m est p fois continûment différentiable dans $\Omega \times (t>0)$, alors u et $\partial u/\partial x_i$ ($i=1, \cdots, n$) sont p fois continûment différentiables dans $\Omega \times (t>0)$; si u_0 est dans $L^2(\Omega)$ et est continue dans Ω, alors $u(x, t) \to u_0(x)$ lorsque $t \to 0$. Les démonstrations reposent sur des formules simples de représentation des solutions (obtenus à l'aide de paramétrix). *J. L. Lions.*

Cinquini Cibrario, Maria. Equazioni a derivate parziali di tipo misto. Rend. Sem. Mat. Fis. Milano **25** (1953–54), 18–40 (1955).

L'A. riassume lo sviluppo della teoria delle equazioni di tipo misto; ricordati i risultati del Tricomi relativi all'equazione detta ora appunto „di Tricomi", l'A. espone la propria classificazione delle equazioni di tipo misto e i principali risultati da lei ottenuti per tali equazioni; espone poi brevemente le ricerche successive, in particolare le piu recenti. *Author's summary.*

Davis, Robert B. Asymptotic solutions of the first boundary value problem for a fourth-order elliptic partial differential equation. J. Rational Mech. Anal. **5** (1956), 605–620.

Let R be a bounded plane region with boundary B. The differential equation problem considered is $L_\varepsilon u_\varepsilon = (\varepsilon^2\Delta+a)(\Delta+b)u_\varepsilon=c$ in R, $u_\varepsilon=f$, $\partial u_\varepsilon/\partial n=g$ on B, where a and b are negative constants, c, f and g given functions. It is shown that, if all the data are sufficiently smooth, $u_\varepsilon=u_0+\bar{\Omega}$, where u_0 is the solution of the problem $L_0u_0=a(\Delta+b)u_0=c$ in R, $u_0=f$ on B and $\bar{\Omega}$ becomes small with ε in the sense that

$$\iint_R \bar{\Omega}^2 dxdy=O(\varepsilon^2),\ \iint_R (\bar{\Omega}^2{}_x+\bar{\Omega}^2{}_y)dxdy=O(\varepsilon^2).$$

The method of proof is an adaptation of one introduced by N. Levinson [Ann. of Math. (2) **51** (1950), 428–445; MR **11**, 439]. Its central part is the construction of a function $\Phi(x, y, \varepsilon)$ for which $L_\varepsilon\Phi=O(\varepsilon)$ in R, $\Phi=0$ on B, $\partial\Phi/\partial n=\partial(u_\varepsilon-u_0)/\partial n$ on B. By means of Green's formula it then follows from the differential equation that

$$\iint_R (u_\varepsilon-u_0-\Phi)^2 dxdy=O(\varepsilon^2).$$

Similarly, the first partial derivatives of $u_\varepsilon-u_0-\Phi$ are small with ε in the same mean square sense. The smallness of $\bar{\Omega}=u_\varepsilon-u_0$ and of its derivatives is then proved as in Levinson's paper.

If a and b are no longer assumed to be constants, and if the boundary conditions on u_ε are replaced by $u_\varepsilon=0$ on B, $\Delta u_\varepsilon+bu_\varepsilon=0$ on B, it is proved by similar methods that $u_\varepsilon-u_0=O(\varepsilon)$ in every closed subdomain of R.

W. Wasow (Los Angeles, Calif.).

Royster, W. C.; and Conte, S. D. Convergence of finite difference solutions to a solution of the equation of the vibrating rod. Proc. Amer. Math. Soc. **7** (1956), 742–749.

The authors deal with two finite-difference simulations of the equation $U_{xxxx}+U_{tt}=0$ and consider the problem associated with the boundary conditions $U(0, t)=U(1, t)=U_{xx}(0, t)=U_{xx}(1, t)=0$ and the initial conditions $U(x, 0)=f(x)$, $U_t(x, 0)=0$. Proceeding in a manner similar to that of W. W. Leutert [J. Math. Phys. **30** (1952), 245–251; MR **13**, 783] relative to the heat-flow equation, they display a sequence of functions $v_M(x, t)$ such that v_M satisfies the difference equation under consideration when M difference intervals are taken in the x direction and also satisfies the boundary conditions as well as the initial gradient condition, and establish its convergence to the solution of the true problem as $M \to \infty$, with the mesh ratio held con-

stant, under very general conditions. As in the reference cited, however, the condition $v_M(x, t)=f(x)$ is not imposed at the initial mesh points but is replaced by a requirement which depends upon M and is not readily interpreted in terms of an actual numerical solution which presumes ignorance of the true solution. Thus the important practical case in which this condition is imposed, and also the gradient condition is simulated, is not treated.

F. B. Hildebrand (Cambridge, Mass.).

Constantinescu-Cătunești, S. Plaques obliques. Une méthode de calcul numérique. Com. Acad. R. P. Romîne **5** (1955), 913–922. (Romanian. Russian and French summaries)

The fourth order partial differential equation of an oblique plate of constant slope is treated by means of two second order difference operators. The resulting system of algebraic linear equations is solved by relaxation. Concentrated and asymmetric loads are considered and a numerical example is discussed. *H. F. Bückner.*

Babič, V. M. On a class of generalized functions. Vestnik Leningrad. Univ. **11** (1956), no. 7, 94–105. (Russian)

Par utilisation de la théorie des distributions [Schwartz, Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; MR **12**, 31, 833], l'A. donne diverses formules du type "Stokes". Applications à l'opérateur

$$\square=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2(x, y, z)}\frac{\partial^2}{\partial t^2}.$$

J. L. Lions (Nancy).

See also: Douglas and Gallie, p. 1241; Teodorescu, p. 1247; Nikitin, p. 1248; Philipzik, p. 1252; Tersenov, p. 1255.

Difference Equations, Special Functional Equations

Hahn, Wolfgang. Zur Stabilität der Lösungen von linearen Differential-Differenzengleichungen mit konstanten Koeffizienten. Math. Ann. **131** (1956), 151–166.

The author considers a differential-difference equation of the form

$$\sum_{j=0}^{k}\sum_{i=0}^{p_j}a_{ij}u^{(i)}(x-h_j)=F(x),$$

x real, $h_j>0$, and shows that, under a "stability" assumption concerning the location of the characteristic roots, every solution has an expansion in the form of a sum of exponentials, with perhaps polynomial coefficients, this expansion being uniformly convergent for $x>0$. Results of similar type have been given by E. M. Wright [Proc. Roy. Soc. Edinburgh. Sect. A. **62** (1949), 387–393; MR **11**, 182] and by R. Bellman and J. M. Danskin [A survey of the mathematical theory of time-lag, retarded control, and hereditary processes, RAND, R-256, 1954].

In all treatments, the Laplace Transform, or equivalent contour integrals, is the main tool. *R. Bellman.*

See also: Gumenyuk, p. 1254.

Integral Equations, Equations in Infinitely Many Variables

Müller, Claus. A new method for solving Fredholm integral equations. Comm. Pure Appl. Math. **8** (1955), 635–640.

The author considers equations in Hilbert space of the form

$$y=Lx=x-Kx, \tag{1}$$

where K is a completely continuous linear operator or, more generally, L is a bounded linear operator with the following property: if T is either L or its adjoint L', and (x_n) is a bounded sequence of elements such that $Tx_n\to y$, then there is an element x such that $Tx=y$, and x is the limit of a subsequence of (x_n).

The principal result is as follows. Let y be an element orthogonal to every ψ for which $L'\psi=0$; then (1) has a unique solution x such that $(x, \varphi)=0$ whenever $L\varphi=0$; furthermore, x can be calculated by the following iterative process: let

$$y_0=y,\ \lambda_n=\frac{\|L'y_{n-1}\|^2}{\|LL'y_{n-1}\|^2},\ z_n=\lambda_nL'y_{n-1},$$

$$y_n=y_{n-1}-Lz_n,\ x_n=\sum_{\nu=1}^{n}z_\nu;$$

then (x_n) is convergent to a limit x with the above properties. *F. Smithies* (Cambridge, England).

Tsuji, Masatsugu. On an integral equation with a kernel $K(x, y)\subset L^p(1<p\leqq 2)$. Jap. J. Math. **23** (1953), 1–14 (1954).

The following results are proved. Let $1<p\leqq 2$, and let $q=p/(p-1)$. Let $K(x, y)$ be measurable in the square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, and suppose that

$$\int_0^1\int_0^1|K(x, y)|^p\,dxdy<\infty,$$

$$\int_0^1\Big(\int_0^1|K(x, y)|^p dy\Big)^{q/p}dx<\infty,$$

$$\int_0^1\Big(\int_0^1|K(x, y)|^p dx\Big)^{p/q}dy<\infty.$$

Then the Fredholm theorems hold for the integral equation

$$\varphi(x)-\lambda\int_0^1K(x, y)\varphi(y)dy=f(x),$$

its associated homogeneous equation

$$\varphi(x)-\lambda\int_0^1K(x, y)\varphi(y)dy=0,$$

and the conjugate homogeneous equation

$$\varphi(x)-\lambda\int_0^1K(y, x)\varphi(y)dy=0,$$

where $f(x)$ and $\varphi(x)$ are required to belong to the space $L^q(0, 1)$, and equality is interpreted as equality almost everywhere.

Similar results are proved for the space $L(0, 1)$ when $K(x, y)$ is bounded, and when it is of the form $|x-y|^{-\alpha}H(x, y)$, where $H(x, y)$ is continuous and $0<\alpha<1$.

[Reviewer's note. The author does not seem to be acquainted with the reviewer's paper in Ann. of Math. (2) **38** (1937), 626–630; see also A. C. Zaanen, Linear analysis, Interscience, New York, 1953, ch. 11, 13; MR **15**, 878].

F. Smithies (Cambridge, England).

Pekeris, C. L. Solution of an integral equation occurring in impulsive wave propagation problems. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 439–443.

In an earlier paper [same Proc. **41** (1955), 629–639; MR **17**, 320] dealing with the propagation of a buried seismic pulse in a uniform elastic half space the author applied operational methods to solve the integral equation

$$(1)\quad p\int_0^\infty e^{-pt}W(t, r, H)dt=$$

$$\frac{p}{c}\int_0^\infty J_0\Big(\frac{p}{c}rx\Big)f(x)\exp\Big[-\frac{p}{c}H\sqrt{(x^2+a^2)}\Big]xdx$$

for W. In (1), $p\equiv\partial/\partial t$; c, r, H are physical parameters; J_0 is the Bessel function of order zero, and $f(x)$ is given.

In the present paper the results of this work are summarized "to facilitate their application to related wave propagation problems." Solutions are also given to another equation similar to (1), from the earlier paper.

J. F. Heyda (Indianapolis, Ind.).

Harazov, D. F. Some questions of the spectral theory of operators depending meromorphically upon a parameter. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **21** (1955), 145–168. (Russian)

In the first part of the paper the author considers equations in Hilbert space of the form

$$(1)\qquad x=A_0x+\lambda\sum_{j=1}^{p}\sum_{k=1}^{m_j}(\lambda-a_j)^{-k}A_{kj}x,$$

where $a_1, \cdots, a_p$ are real numbers, $a_1<a_2<\cdots<a_p$, A_0 and the A_{kj} are completely continuous linear operators, and there is a bounded positive self-adjoint operator H such that HA_0 and the HA_{kj} are all self-adjoint. Various sets of sufficient conditions are found for the existence of a real characteristic value of (1); a typical set is (i) $a_j\neq 0$ $(1\leqq j\leqq p)$, (ii) A_0 has no positive characteristic value $\leqq 1$, (iii) some of the a_j are negative and, if a_r is the algebraically greatest of these, then (iv) $A_{m,r}$ has a negative characteristic value. Results on the distribution of the characteristic values are also obtained; when all $m_j=1$, these give conditions under which the characteristic values are all real.

In the second part some of these results are extended to equations of the form

$$x=\sum_{n=0}^{m}\lambda^nA_nx+\lambda\sum_{i=1}^{\infty}(\lambda-a_i)^{-1}B_ix,$$

where the a_i are all real, $\sum|a_i|^{-1}<+\infty$, the B_i are uniformly bounded, and all the operators A_n and B_i are completely continuous and simultaneously symmetrisable in the same sense as in the first part.

In the last two parts more far-reaching results are obtained for a class of equations of the form

$$x=y+A_0x+\lambda\{A_1x+\lambda\sum_{i=1}^{p}(\lambda-a_i)^{-1}H_ix\},$$

where all the a_i are real and non-zero, and the H_i are self-adjoint operators of finite rank. Under a set of conditions, too complicated to be detailed here, it is shown by a variational method that the characteristic values are all real, and can be arranged as a sequence (λ_n) such that $|\lambda_1|\leqq|\lambda_2|\leqq\cdots$ and $|\lambda_n|\to\infty$. The corresponding characteristic vectors can be made orthonormal in an appropriate sense, and a series of expansion theorems in terms of weak convergence is obtained. *F. Smithies.*

Bykov, Ya. V. On the theory of linear integro-differential equations. Dokl. Akad. Nauk Uzbek. SSR. **1953**, no. 8, 3–6. (Russian. Uzbek summary)

A series of existence theorems without proofs concerning the system

$$\frac{\partial}{\partial x}z_i(x)+\sum_{k=1}^{n}a_{ik}(x)z_k(x)-\lambda\int_a^x\sum_{j=1}^{n}K_{ij}(x, t)z_j(t)\,dt=0$$

given initial conditions $z_i(a)=b_i$, where a_{ik} are continuous near a and $K_{ij}(x, t)$ are possibly discontinuous kernels as in the usual theory of integral equations.

D. C. Kleinecke (Albuquerque, N.M.).

Bykov, Ya. V. On a class of integro-differential equations. Kirgiz. Gos. Univ. Trudy Fiz.-Mat. Fak. **1953**, no. 2, 85–109. (Russian)

A much more detailed account of the methods reported in another paper by the author [Dokl. Akad. Nauk UzSSR **1953**, no 6, 3–6] for solving the integro-differential equation

$$L(y)+\int_a^x\sum_{j=1}^{k}P_j(x-t)\cdot\exp\beta_j(t-x)M(y(t))\,dt=0,$$

where L and M are homogeneous linear differential operators with constant coefficients, $P_j(u)$ are polynomials in u and β_j are constants. Some related problems are also considered. *D. C. Kleinecke.*

See also Birger p. 1206; Babuška et al, p. 1253.

Calculus of Variations

Matveev, N. M. On Lichtenstein's sufficient condition for a weak minimum. Leningrad. Gos. Univ. Uč. Zap. **137**. Ser. Mat. Nauk **19** (1950), 204–220. (Russian)

The author extends, to variational problems

$$\delta\int_a^b F(x, y, z, y', z')dx=0$$

involving two unknown functions $y=y(x)$ and $z=z(x)$ with variable end points, a sufficient condition for a weak minimum due to Lichtenstein [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1919**, 161–192]. *R. Finn.*

Malinskiĭ, K. K. On derivation of necessary conditions in a variational problem with movable boundary for double integrals. Leningrad. Gos. Univ. Uč. Zap. **137**. Ser. Mat. Nauk **19** (1950), 198–203. (Russian)

Derivation of the nth order variation of

$$J=\iint_{(\Omega)}F(x, y, z, p, q)dxdy$$

taking into account variations of the boundary curve.

R. Finn (Pasadena, Calif.).

Vâlcovici, Victor. Sur les principes de Hamilton et de la moindre action. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 441–448 (1956).

Infinitely many variational principles are obtained of which Hamilton's principle and the principle of least action may be regarded as special cases. This is done by comparing the actual motion to a varied (but not necessarily synchronous) motion. In the case of Hamilton's principle, we do have synchronism, and this fact may be regarded as a restriction on the varied motion. In the

principle of least action, the varied motion is not required to be synchronous; instead it is restricted to the requirement of constant energy. In general, it is possible to obtain other variational principles by introducing other restrictions of this type on the varied motions. These restrictions may be written as integral conditions on the variation δt of the time. *D. C. Lewis.*

Storchi, Edoardo. Condizioni caratteristiche per la validità dei principî di Hamilton e di Hölder. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 162–167.

The author specifies the widest class of varied motions within which the requirement that a given "action" integral should be stationary yields the mechanically correct motion. Thus, for example, Hamilton's familiar principle,

$$\delta\int_{t_0}^{t_1}(T-U)dt=0,$$

is valid not only for synchronous variations (for which $\delta t=0$) but for all variations for which

$$\int_{t_0}^{t_1}(\cdot\ +U)\frac{d}{dt}(\delta t)dt=0.$$

A similar result is obtained relative to Hölder's least action principle as well as to any arbitrary principle anyone might propose. The restrictions on the varied motions are always in the form of integral conditions.
D. C. Lewis (Pacific Palisades, Calif.).

See also: Wegner, p. 1253; Vaĭnšteĭn, p. 1258.

Theory of Probability

Münzner, Hans. Zur Frage: Binomialverteilung oder Poissongesetz? Bl. Deutsch. Ges. Versicherungsmath. 2 (1956), 405–412.

In a pure death process [cf. Feller, An introduction to probability theory and its applications, Wiley, New York, 1950; MR **12**, 424] acting on a community of initial size N, the probability distribution of the number of deaths after a given time T is Poisson or binomial according as the probability of a further death at time $t<T$ after n deaths have already occurred is, respectively, independent of n or $N-n$ times a function independent of n. The author formally generalizes both these results by ascribing a probability of death to individual j which depends on the number n. Neither of these generalizations appears to correspond to any conceivable situation in the real world. *H. L. Seal* (New York, N.Y.).

Takács, L. On secondary stochastic processes generated by recurrent processes. Acta Math. Acad. Sci. Hungar. 7 (1956), 17–29. (Russian summary)

English version of Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 187–197; MR **17**, 276. *E. Lukacs.*

Fréchet, Maurice. Sur les tableaux de corrélation dont les marges sont données. C. R. Acad. Sci. Paris **242** (1956), 2426–2428.

In a previous paper [Ann. Univ. Lyon. Sect. A. (3) 14 (1951), 53–77; MR **14**, 189] the author found distribution functions which are the upper and lower limits of the joint distribution functions of a pair of random variables, given their individual distributions. He now proceeds to draw simple conclusions from this result. For example, any distribution function between the two limits has the given one-dimensional marginal distributions.
J. L. Doob (Geneva).

Ciucu, Gh. La loi des grands nombres pour les variables aléatoires liées. Com. Acad. R. P. Romîne **5** (1955), 1253–1256. (Romanian. Russian and French summaries)

For each n, let $x_{n1}, \cdots, x_{nn}$ be random variables with zero means, and suppose that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\mathfrak{M}^{i}\{\mathfrak{M}_{S_{(n,i-1)}}{}^{2}(x_{ni})\}=0,$$

where $S_{(n,s)}=\sum_{j=1}^{s}x_{nj}$, and the random variable in the subscript conditions the indicated expectation. The author proves, generalizing a result of Loève [J. Math. Pures Appl. (9) **24** (1945), 249–318; MR **7**, 458 that l.i.m. $_{n\to\infty}S_{(n,n)}=0$ if $\lim_{n\to\infty}\sum_{i=1}^{n}\mathfrak{M}(x_{ni}{}^{2})=0$.
J. L. Doob (Geneva).

Sapogov, N. A. Law of the iterated logarithm for sums of dependent quantities. Leningrad. Gos. Univ. Uč. Zap. **137**. Ser. Mat. Nauk **19** (1950), 160–179. (Russian)

The author proves results announced in earlier papers [Dokl. Akad. Nauk SSSR (N.S.) **63** (1948), 353–356, 487–490; MR **10**, 310, 384] and develops them further. That is, he proves that the law of iterated logarithm is valid for sequences of random variables satisfying the conditions of Bernstein's classical central limit theorem for dependent random variables, and applies this result. He proves that the law of iterated logarithm is applicable to Markov processes under various hypotheses, weaker than those imposed by Doeblin [Publ. Fac. Sci. Univ. Masaryk no. **236** (1937), 3–13] and Sarymsakov and Sultanova [Dokl. Akad. Nauk SSSR **59** (1948), 1249–1252; MR **9**, 451]. For example, it is sufficient if the mth transition probability is given by a density $\phi_m>a_1$, if the mth random variable involved is bounded in modulus by a_2 and has oscillation $\geq a_3$, where the indicated constants are positive and independent of m.
J. L. Doob (Geneva).

Finkel'šteĭn, B. V. Limiting distribution of the terms of a variational series of quantities related by a stationary Markov chain. Ukrain. Mat. Ž. **7** (1955), 313–332. (Russian)

The author apparently bases his work on the (false) notion that, if $x_1, \cdots, x_n$ are random variables forming a Markov chain, then $f(x_1), \cdots, f(x_n)$ also form a Markov chain. *J. L. Doob* (Geneva).

Skorohod, A. V. On a class of limit theorems for Markoff chains. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 781–784. (Russian)

For each n, let $0=\xi_{n0}, \cdots, \xi_{nn}$ be random variables forming a Markov process with stationary transition probabilities, and define $\xi_n(t)=\xi_{n[t(n+1)]}$, for $0\leq t<1$, where $[\alpha]$ is the integral part of α. Let $\{\xi(t), 0\leq t\leq 1\}$ be a Markov process with stationary transition probabilities. It is stated that if, when $n\to\infty$, the $\xi_n(t)$ process transition probabilities go to the $\xi(t)$ process transition probabilities in a carefully stated sense, then the distribution of a functional defined on the $\xi_n(t)$ process sample functions converges to that of the functional on the $\xi(t)$ process

sample functions. The $\xi(t)$ process is of mixed diffusion discrete type. The functional F is defined, and continuous almost everywhere, on the space of right-continuous functions on [0, 1] with left-hand limits existing at all points, the space being provided with an appropriate topology and with the measure induced by the $\xi(t)$ process probabilities. *J. L. Doob* (Geneva).

Castoldi, Luigi. Irreversibilità delle catene ordinarie di Markov. Boll. Un. Mat. Ital. (3) **11** (1956), 16–21.

Under very general conditions, a Markov chain leads to an asymptotic probability distribution of its states; the author shows that this is not true of the chain obtained by reversing the direction of time. *S. K. Zaremba.*

Poliščuk, E. M. The mean value and integral of a functional. Ukrain. Mat. Ž. **8** (1956), 59–75. (Russian)

Let a, b be continuous functions on [0, 1] with $a(t)<b(t)$, and let C be the class of continuous functions x on [0, 1], with $a(t)<x(t)<b(t)$. Let F be a function on C which can be expressed as a limit of the form

$$\lim_{n\to\infty} \Phi_n[x(t_1^n), \cdots, x(t_{k_n}^n)],$$

where $t_i^n \leqq t_{i+1}^n$ and $\max_i (t_{i+1}^n - t_i^n) \to 0$. Then the integral of F on C is defined as the limit (if it exists) of the integral of Φ_n over the k_n dimensional interval defined by $a(t_i^n) \leqq x_i^n \leqq b(t_i^n)$, $i \leqq k_n$. The mean value MF of F is defined as the limit (if it exists) of the average value of Φ_n on this interval. These quantities are evaluated in various cases. For example, MF is found when F is given by

$$\int_0^1 \cdots \int_0^1 H[x(t_1), \cdots, x(t_p); t_1, \cdots, t_p]\, dt_1 \cdots dt_p.$$

In an appropriate sense it is shown that, in this example, $F=MF$ almost everywhere on C, and the integral of F has the form const MF, where the constant is arbitrary, depending on the choice of the t_j^n. *J. L. Doob* (Geneva).

Onoyama, Takuji. Linear translatable stochastic functional equations and random distributions. Mem. Fac. Sci. Kyusyu Univ. Ser. A. **10** (1956), 29–43.

Let $Y(\phi)$ be a specified stationary random distribution, that is, a random variable (in L_2) valued function, defined on Schwartz' space of real-valued C_∞ functions with compact supports on the real line, such that the covariance function is invariant under translations. The object of the paper is to treat the solution of

$$(*) \qquad \Lambda X(\phi) = \int_0^1 X[\phi(t-h)]\, d\varphi(h) = Y(\phi),$$

where φ is of bounded variation on [0, 1], and X is a random distribution. The solution is obtained in terms of the random Cauchy series introduced by Kitagawa [Jap. J. Math. **22** (1952), 1–18; MR **16**, 151], and depends on the character of the function G defined by $G(\lambda) = \int_0^1 e^{\lambda t}\, d\varphi(t)$. If $X(t)$ comes from a process with stationary kth order increments, and if $Y(\phi) \equiv 0$, then terms of the Cauchy series corresponding to certain zeroes of G vanish. *J. L. Doob* (Geneva).

Takács, Lajos. On a probability treatment of electron-tube plate-current oscillation. Magyar Tud. Akad. Mat. Fiz. Tud. Oszt. Közl. **6** (1956), 27–51. (Hungarian)

The author proposes a stochastic model for the random fluctuation of the plate current in a diode. Formulae are derived for the moments and for the characteristic function of the instantaneous current, for the variance of the average current and for the spectrum of the plate current. The discussion of the model is based on the author's method given in an earlier paper [same Közl. **4** (1954), 473–504; MR **16**, 723]. *E. Lukacs.*

Mathematical Statistics

★ **Walsh, John E. A Monte Carlo technique for obtaining tests and confidence intervals for insurance mortality rates.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 265–277. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

The author proposes a method for obtaining confidence intervals for mortality rates. The arguments are heuristic and somewhat vague. *E. Lukacs* (Washington, D.C.).

Derman, Cyrus. An application of Chung's lemma to the Kiefer-Wolfowitz stochastic approximation procedure. Ann. Math. Statist. **27** (1956), 532–536.

Let $M(x)$ be a regression function which is strictly increasing for $x<\theta$ and strictly decreasing for $x>\theta$. Under certain conditions it is shown that the stochastic approximation process described in same Ann. **23** (1952), 462–466 [MR **14**, 299] converges stochastically to θ. Under certain additional conditions the distribution of the normalized process is shown to converge to a normal distribution. The method is one used by Chung [ibid. **25** (1954), 463–483; MR **16**, 272]. *J. Wolfowitz.*

Kuznecov, P. I.; and Stratonovič, R. L. On the mathematical theory of correlated random points. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 167–178. (Russian)

The authors consider random points x in a Euclidean space S: their formulae are stated for one dimension, but are interpretable also for several dimensions. They assume that the probability of getting at least one of the random points in each of the intervals $(x_1, x_1+dx_1), \cdots, (x_s, x_s+dx_s)$ is equal to

$$f_s(x_1, \cdots, x_s)dx_1 \cdots dx_s\{1+O(\Delta)\},$$

where $\Delta = \max(dx_1, \cdots, dx_s)$ and the functions f_s ($s=1, 2, \cdots$) are given. They introduce "correlation distribution functions" g_s by the following identity between (formal) power series:

$$\sum_{s=0}^{\infty} \frac{1}{s!} \sum_{\mu_1, \cdots, \mu_s} f_s(x_{\mu_1}, \cdots, x_{\mu_s}) z_{\mu_1} \cdots z_{\mu_s} =$$

$$\exp\left\{ \sum_{s=1}^{\infty} \frac{1}{s!} \sum_{\mu_1, \cdots, \mu_s} g_s(x_{\mu_1}, \cdots, x_{\mu_s}) z_{\mu_1} \cdots z_{\mu_s} \right\},$$

where $f_0=1$. Supposing $u(x)$ to be an arbitrary function, R a fixed finite interval in S, and $\bar{x}_1, \cdots, \bar{x}_n$ to be the random points that fall into R, they obtain for the expectation $L_R\{u\}$ of $\prod_{i=1}^{n} \{1+u(\bar{x}_i)\}$ the expression

$$\sum_{s=1}^{\infty} \frac{1}{s!} \int_R \cdots \int_R f_s(x_1, \cdots, x_s) u(x_1) \cdots u(x_s)\, dx_1 \cdots dx_s,$$

and for $\log L_R\{u\}$ the corresponding expression with f_s replaced by g_s. In particular, they find that the characteristic function $\theta(t)$ of the distribution of n is given by

$$\theta(t) = \exp\{\sum_{1}^{\infty} (e^{it}-1)^s T_s/s!\},$$

where T_s is the (s-fold) integral of $g_s(x_1, \cdots, x_s)$ over $R\times\cdots\times R$. If the random points are independent, then $g_2=g_3=\cdots=0$ and $\theta(t)$ corresponds to a Poisson distribution. The authors suggest that in many applications (for example, to statistical physics) the g_s's could be neglected except for small values of s, but that in the formulae in terms of the f_s's a similar neglect would be less satisfactory. [Remark: in the special case where T_s equals the sth moment of some bounded positive random variable the function $\theta(t)$ will correspond to a type of "contagious" distribution considered by J. Neyman, Ann. Math. Statist. **10** (1939), 35–57, esp. § 3.]

H. P. Mulholland (Birmingham).

Fraser, D. A. S. A vector form of the Wald-Wolfowitz-Hoeffding theorem. Ann. Math. Statist. **27** (1956), 540–543.

A vector form of a theorem by Hoeffding [same Ann. **22** (1951), 558–566; MR **13**, 363] and Wald and Wolfowitz [ibid. **15** (1944), 358–372; MR **6**, 163].

J. Wolfowitz (Ithaca, N.Y.).

Haldane, J. B. S. The Wilcoxon and related tests of significance. Experientia **12** (1956), 205.

Goodman, Thomas P. Technique for approximate measurement of correlation coefficients. J. Appl. Phys. **27** (1956), 773–775.

Whittle, P. The statistical analysis of a seiche record. J. Marine Res. **13** (1954), 76–100.

The author analyzes a record of the water level in a rock channel on the Wellington coast, to test and illustrate certain methods of analysis, notably the use of the correlogram and the periodogram. He finds that these tools complement each other for detecting respectively continuous-spectrum and line-spectrum type motions. By admitting line spectra in his statistical model he elicits more information from the data than by the pure continuous-spectrum analysis. Indeed, his data would support a further step in this direction. Thus his wave of period 10.87 units occur in well-marked trains bearing little phase relation to one another. The changes in phase represent cancellations occurring in the computation of the periodogram, by which that statistic is rendered weaker than could be obtained by giving more individual attention to the trains of waves.

In addition to this analysis, the author notes physical features of the coast, notably a concave reflecting shore, which account for substantial parts of the variation. In such cases it is possible to construct a model which yields a much more complete representation of the motion.

A. Blake (Royal Oak, Mich.).

Groves, Gordon W. Numerical filters for discrimination against tidal periodicities. Trans. Amer. Geophys. Union **36** (1955), 1073–1084.

The author computes a family of nineteen symmetric sequences of weighting coefficients for use in suppressing known components of the tide in order to make apparent movements of period about two days or longer. The sequences in the family vary in length from fifteen to fifty-one terms (hours). For each sequence length, the criterion used for determination of the coefficients is minimization of the square of the output of the filter when applied to a prescribed standard tide plus noise due to rounding. By use of the weighting coefficients it is possible to recognize changes of as little as a few millimeters occurring in a few days. Choice of the length of the sequence for any particular application is made by considering computational convenience, the period of the motion in whose favor it is desired to discriminate, and departure of the tide at the particular place from the standard. The last of these factors could be exploited to obtain an extension of the family of sequences of coefficients.

A. Blake (Royal Oak, Mich.).

Huffman, David A. A study of the memory requirements of sequential switching circuits. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. **293** (1955), 28 pp.

In this report sequential switching circuits are viewed as combinational circuits plus delay (memory) devices connected from the output to the input, and the size of this memory unit for arbitrary asynchronous sequential switching circuits is studied. The central idea is that of the intermeshed row-set.

The use of Hamming error-correcting codes is discussed, and is shown to be effective to synchronous circuits, but of limited practicability in asynchronous circuits.

R. W. Hamming (Murray Hill, N.J.).

Elias, Peter. Error-free coding. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. **285** (1954), 12 pp.

"This report describes constructive procedures for encoding messages to be sent over noisy channels so that they may be decoded with arbitrarily low error rate. The procedures are a kind of iteration of simple error-correcting codes, such as those of Hamming and Golay; ···". The procedures permit the transmission of error-free information at a positive rate and have the two properties: (a) they are systematic, and (b) the receiver can make the error probability as low as he desires, provided he accepts enough delay in the decoding.

It is also shown that there exist codes of this error-free character which transmit information at rates arbitrarily near the channel capacity.

R. W. Hamming.

Guest, P. G. Grouping methods in the fitting of polynomials to unequally spaced observations. Biometrika **43** (1956), 149–160.

This is a sequel to a previous article concerning equally spaced observations [Biometrika **41** (1954), 62–76; MR **15**, 991]. In this article, the author must consider the biases as well as the efficiencies of grouped estimates. These are shown to depend largely on two measures of departure from uniform spacing, a mesure of crowding towards one end of the range of the independent variable and a measure of crowding towards the center. Tables are given to indicate the efficiency of grouping for various values of these measures for the number of groups (N) equal to 5, 9, 12, 16 and 21. Simple methods are given to enable the experimenter to obtain approximate estimates of bias if his data are grouped. An example is presented for $N=17$ with 67 total observations. Step function procedures are also presented for unequally spaced observations. A time study was made on the various computing methods for the above example.

R. L. Anderson (Raleigh, N.C.).

Teicher, Henry. Identification of a certain stochastic structure. Econometrica **24** (1956), 172–177.

Let $\{U_i\}$ be independent, identically distributed, chance

variables. Let

$$X_t=U_t+\alpha U_{t-1} \quad (t=1, 2, \cdots),$$

where α is a constant, and let $\varphi(t)$ be the characteristic function (c.f.) of U_t. Let U (or X) be a chance variable with the same distribution as U_t (or X_t). Either α or $\varphi(t)$ is said to be identifiable if it is uniquely determined by the distribution of $X_1, X_2, \cdots$. (In the results to be cited only the distribution of (X_1, X_2) seems to come into play.) Theorem 1: A sufficient condition that α be identifiable is that the distribution of U (or X) be non-normal. Theorem 2: If the distribution of X (or U) is normal, a necessary and sufficient condition that both α and $\varphi(t)$ be identifiable is that α ranges over a set A of real values such that for $\alpha\neq 1$, $\alpha\in A$ implies $(1/\alpha)\notin A$. Theorem 3: If the c.f. of X is non-normal and not identically zero in any (non-degenerate) interval and A does not contain -1 as an element, then α and $\varphi(t)$ are identifiable.

A linear structure relationship is also examined briefly. *J. Wolfowitz* (Ithaca, N.Y.).

Barton, D. E. A class of distributions for which the maximum-likelihood estimator is unbiased and of minimum variance for all sample sizes. Biometrika **43** (1956), 200–202.

If, in

$$p(x, \theta)=\exp\{\alpha(\theta)f(x)+\beta(\theta)+g(x)\},$$

one reparametrizes with $\varphi=-\beta'(\theta)/\alpha'(\theta)$, the maximum likelihood estimator of φ is unbiased with a variance equal to that specified by the Fréchet-Darmois-Cramér-Rao lower bound. The multivariate multiparameter analogue is also discussed. *H. Teicher.*

de Munter, P. Fonction de puissance de certains tests du caractère aléatoire d'un échantillon. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 291–312.

Au § 2 nous étudions les tests-S_k, généralisation d'un test-S considéré par Mann. Au § 3 nous étudions le test-T de Mann. Au § 4 nous étudions les test-D et -S de Foster et Stuart. L'étude des tests porte sur la consistance, la valeur de la fonction de puissance (pour des alternatives bien spécifiées) et l'efficience. Au § 5 les tests sont comparés entre eux et nous avons calculé leur efficacité asymptotique relative. [From the author's summary. S_k is the number of negative differences $X_{i+j}-X_i$ for $i\leq n-k$ and $1\leq j\leq k$, where $X_1, \cdots, X_n$ are the random variables upon which the test is based.] *J. Kiefer.*

Chanda, K. C. On some aspects of non-regular testing of hypotheses. Calcutta Statist. Assoc. Bull. **6** (1955), 95–98.

For a population $f(x, \theta)=p(x)/g(\theta)$, $a(\theta)\leq x\leq b\leq\infty$ and zero elsewhere with $a(\theta)$ continuous and monotone increasing, tests of a simple hypothesis H_0: $\theta=\theta_0$ versus alternatives on one or both sides are discussed. Curiously, the author requires that the critical region w_n for H_0 be a subset of the spectrum, i.e., of $[a(\theta_0), b]$ even for alternatives $\theta<\theta_0$, where the power (but not the size) may be increased by adjoining $[-\infty, a(\theta_0)]$ to w_n. This leads him to the erroneous conclusion that no unbiased size α test exists in the case $\theta<\theta_0$. The distributions in question possess monotone likelihood ratios from which many of the conclusions flow. *H. Teicher* (Lafayette, Ind.).

Doornbos, R.; and Prins, H. J. Slippage tests for a set of gamma-variates. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 329–337.

A reprinting of the paper reviewed in MR **17**, 641.

Lawley, D. N. Tests of significance for the latent roots of covariance and correlation matrices. Biometrika **43** (1956), 128–136.

The author gives a number of large sample tests for hypotheses involving the latent roots of sample covariance and correlation matrices that arise in factor analysis. More precisely, the tests are of the hypothesis that the $n-k$ smallest latent roots are equal, it being assumed that the effects of the k largest roots have been removed. The tests are based on the asymptotic theory of maximum likelihood ratio tests, but following Bartlett [J. Roy. Statist. Soc. Ser. B. **16** (1954), 296–298; MR **16**, 1039], he gives multipliers that yield a "better approximation" in the sense that the moments then agree with those of χ^2 to quantities of order $(1/n^2)$. The determination of the correct multiplier requires some rather heavy matrix algebra; also the final expression involves unknown parameters that must be estimated from the sample.

D. G. Chapman (Seattle, Wash.).

Geffroy, Jean. Sur une propriété de l'écart maximum entre les fonctions de répartition théorique et empirique d'un échantillon de n points à deux dimensions. C. R. Acad. Sci. Paris **242** (1956), 2283–2285.

It is shown that the maximum deviation of a cumulative distribution function (c.d.f.), say F, from its sample analogue, the so-called empirical c.d.f., is not distribution free in the two dimensional case even for F absolutely continuous. *H. Teicher* (Lafayette, Ind.).

Medhi, J. A note on the risks of error involved in the sequential ratio test. Biometrika **43** (1956), 231–234.

The author computes the probability of stopping and making the wrong decision after one, two, or three observations in testing which of two values is the mean of a normal distribution, in order to obtain illustrations of how conservative Wald's lower bounds on the probability of making an incorrect decision can be. *J. Kiefer.*

Bartholomew, D. J. A sequential test of randomness for events occurring in time or space. Biometrika **43** (1956), 64–78.

A method has been given for testing whether a sequence of events is occurring at random in time or space when the alternative is a trend. The test is derived on the assumption that the alternative can be represented by a function of the form $\lambda(T)=\mu(\mu T)^a$ for $-1<a<\infty$. The theory of sequential analysis is used to derive the test together with its operating characteristic and average sample number functions. The application of the test under various circumstances has been discussed from the theoretical standpoint and illustrated on mining accident data. (From the author's summary.) *J. Kiefer.*

★ **Wasow, Wolfgang. On the asymptotic transformation of certain distributions into the normal distribution.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 251–259. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

A procedure introduced for numerical work with the Poisson summation (effectively the χ^2-distribution) is to transform the variable into an asymptotic series in terms of a normal variable. This paper establishes conditions on density functions which permit such asymptotic ex

pansions and proves also that the coefficients of the expansion are polynomials, a result conjectured by Hotelling and Frankel [Ann. Math. Statist. 9 (1938), 87–96].
J. Riordan (New York, N.Y.).

★**Walther, Alwin. Experiments and models for the Monte Carlo method.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 278–282. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

Cox, C. P. Latin-square designs with individual gradients in one direction. Nature 177 (1956), 1092.
Consider an $n \times n$ latin-square design having the underlying mathematical model

$$y_{ij}=\mu+\chi_j+\beta_j a_{ij}+\theta_i+e_{ij},$$

where y_{ij} is the observed value for the experimental unit of the ith treatment in the jth column; μ is the general mean; χ_j and β_j are location and slope parameters for the linear variation in the jth column; a_{ij} is the discrete row constant for the ith treatment in the jth row measured from a zero mean at the centre of the rows, such that

$$\sum_{i=1}^{n} a_{ij}^2=\sum_{j=1}^{n} a_{ij}^2=n;$$

and θ_i is the ith treatment parameter. The author assumes that the e_{ij} is distributed as a normal distribution with mean zero and variance σ^2 and states that unbiased estimates of the treatment comparisons can be obtained with variance-covariance matrix $-n[AA'-n^2]^{-1}$, where $[A]$ is the matrix $[a_{ij}]$.
Reviewer's note: Under the assumptions of the author it is also necessary that the e_{ij} be uncorrelated.
M. Zelen (Washington, D.C.).

Kivéliovitch, M.; et Vialar, J. Quelques nouveaux tests pour l'étude des séries chronologiques. II. J. Sci. Météorol. 6 (1954), 151–166.
[For part I see same J. 6 (1954), 73–83; MR 16, 942.] Given a finite sequence whose ordinates are confined to discrete values. Taking note of the relative frequencies of occurrence of these values and attributing these to the population from which the sequence was drawn, the authors inquire what are the expected frequencies of occurrence of runs up and down of various lengths, and of sequences of successive runs of given lengths, in random samples from such a population. They derive formulas for these frequencies, extending the results of Part I. They give also the characteristic function, the standard error, and an approximate confidence interval. They also proposed two additional types of test for randomness of the given sequence: first, that the foregoing procedures be applied not only to the original sequence but to the sequence of its lengths of runs up and down; second, a test of lengths of sequences of equal values and the relative number of observations that are in these sequences. These latter tests are proposed primarily to reduce the arithmetic work. (In the table on page 165, in the column "Observées Total," the entries 130, 134, and 295 should be 295, 130, and 134 respectively.) *A. Blake.*

Wise, J. Stationarity conditions for stochastic processes of the autoregressive and moving-average type. Biometrika 43 (1956), 215–219.
Given a process x_t defined by $x_t+\alpha_1 x_{t-1}+\cdots+\alpha_k x_{t-k}=\varepsilon_t$, the conditions for x_t to be stationary are converted into determinant form, with explicit formulae for $k=1$, 2, 3, 4. *H. Wold* (Uppsala).

Christie, Lee S.; and Luce, R. Duncan. Decision structure and time relations in simple choice behavior. Bull. Math. Biophys. 18 (1956), 89–112.
The authors present a model for the making of decisions in psychological experiments in terms of a simple stochastic process operating in an oriented linear graph. Explicit formulas for the distribution of reaction time are found in a few very simple cases. *C. A. B. Smith.*

★**Berger, Martin J. An application of the Monte Carlo method to a problem in gamma ray diffusion.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 89–102. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

★**Beach, L. A.; and Theus, R. B. Stochastic calculations of gamma ray diffusion.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 103–122. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

Chapman, Douglas G. Estimating the parameters of a truncated gamma distribution. Ann. Math. Statist. 27 (1956), 498–506.
A table is given to simplify the estimation of the parameters of an incomplete gamma or Type III distribution when using the method of maximum likelihood. A new procedure, similar in some respects to a modified minimum χ^2, is also suggested for estimating the parameters of a truncated gamma distribution. The asymptotic variance-covariance matrix of the estimates is determined. This method is applicable to a number of other truncated distributions, whether the truncation is in the center or in the tails of the distribution. *L. A. Aroian.*

Banerjee, D. P. On the application of operational calculus to the solution of distribution problems. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 437–445.
The Mellin transform is applied to find the distribution of the product and the quotient of two independent variables. This use of the Mellin transform has previously been reported by B. Epstein [Ann. Math. Statist. 19 (1948), 370–379; MR 10, 552]. Applications are made to various likelihood criteria. The distribution of the product of two independent beta distributions is investigated. The many misprints and vague statements of theorems makes it difficult to gauge the author's intentions.
L. A. Aroian (Culver City, Calif.).

Stuart, Alan. The efficiencies of tests of randomness against normal regression. J. Amer. Statist. Assoc. 51 (1956), 285–287.
The asymptotic efficiency in the sense of Pitman of ten tests of randomness against the alternative of normal regression is given. Essentially, this is a supplement to an earlier paper by the same author [same J. 49 (1954), 147–157; MR 15, 728]. *G. E. Noether* (Boston, Mass.).

Freund, John E.; and Pozner, Arthur N. Some results on restricted occupancy theory. Ann. Math. Statist. 27 (1956), 537–540.
The authors consider the consistency of discrete ratings

by k judges using as a model the theory of restricted occupancy, in which all possible distributions of r indistinguishable objects among k cells with at most m objects per cell are assumed equally likely. This provides a test of whether the total r points awarded by the k judges are randomly distributed among the k judges. Related problems were studied by Baticle [C. R. Acad. Sci. Paris **197** (1933), 632–634; **201** (1935), 862–864; **222** (1946), 355–357; MR **7**, 457]. Tables of the restricted occupancy coefficients for values of k from 1 to 20 and r from 1 to 40 were given by Pozner [M. A. Thesis, Virginia Polytech. Inst. 1955]. *H. P. Edmundson.*

Kozelka, Robert M. Approximate upper percentage points for extreme values in multinomial sampling. Ann. Math. Statist. **27** (1956), 507–512.

The author considers a multinomial vector $(F_1, F_2, \cdots, F_k)$ with equal cell probabilities. He obtains an approximation to the MGF of $\max F_i$ for the case $k=3$; for $k \geqq 2$ and $(t_1, t_2, \cdots, t_k)$, the k-variate normal approximation of $(F_1, F_2, \cdots, F_k)$, he gives approximations to the upper 1% and 5% points of $\max t_i$, with a table for $2 \leqq k \leqq 25$. The application is $\max t_i \geqq t^*$ as a test for equal cell probabilities. *R. F. Tate* (Seattle, Wash.).

Theory of Games, Mathematical Economics

Dulmage, Lloyd; and Halperin, Israel. On a theorem of Frobenius-König and J. von Neumann's game of hide and seek. Trans. Roy. Soc. Canada. Sect. III. (3) **49** (1955), 23–29.

John von Neumann [Contributions to the theory of games, vol. 2, Princeton, 1953, pp. 5–12; MR **14**, 998] has introduced the following game of hide and seek in connection with the optimal assignment problem. The hider selects one of the places in an $n \times n$ matrix of real $\alpha_{ij} > 0$ $(i, j = 1, \cdots, n)$ and the seeker selects a row or column of the matrix. If the seeker "finds" the hider in the line selected, he receives α_{ij}; otherwise, he receives nothing. This paper explores the connection between this game and the oft-proved theorem of König-Frobenius which asserts that every term in the expansion of an $n \times n$ matrix vanishes if and only if the matrix contains a $p \times (n+1-p)$ zero submatrix. An elementary proof of this theorem is given and the result is used to determine the (known) optimal strategies of the hider. In a final section, the optimal strategies for the seeker are investigated in some detail. *H. W. Kuhn* (Bryn Mawr, Pa.).

Nikaidô, Hukukane. New aspects of von Neumann's model with special regard to computational problems. Ann. Inst. Statist. Math., Tokyo **6** (1955), 223–230.

In a previous paper [Econometrica **22** (1954), 49–53; MR **15**, 888] the author has given an elementary existence proof for a unique equilibrium for the von Neumann linear expanding model of production along the following lines: Let $A=(a_{ij})$ and $B=(b_{ij})$ be two $m \times n$ non-negative real matrices such that every row of A contains a positive element and $a_{ij}+b_{ij}>0$ for all i and j. For each real number ω, form the matrix game with payoff $B-\omega A$ with value $V(\omega)$. Then the unique solution σ of $V(\omega)=0$ provides the expansion and interest rates of the von Neumann equilibrium. In this paper, it is noted that production processes corresponding to $V(\omega)<0$ are economically impossible, while those corresponding to $V(\omega)>0$ overproduce every good and yield profit to each form in operation. Dynamic aspects (in the form of a differential equation and a difference equation) are introduced in terms of a monetary authority which reacts to alter ω when $V(\omega) \neq 0$. The theorems assert a type of stability in the large for the von Neumann model. The last line in the single numerical example should end: $\sigma = 1.213$. *H. W. Kuhn* (Bryn Mawr, Pa.).

Foulkes, J. D.; Prager, W.; and Warner, W. H. On bus schedules. Management Sci **1** (1954), 41–48.

A bus scheduling problem is analyzed in this paper. As abstracted here, it amounts to the minimization of a quadratic function subject to linear constraints (both equations and inequalities). A simple necessary condition for an efficient schedule is developed and a diagrammatic method is proposed for formulating the equations of this condition directly from the bus network. *H. W. Kuhn.*

Hulthén, Lamek. What is operational analysis? Nordisk Mat. Tidskr. **4** (1956), 87–101, 120. (Swedish. English summary)

Ackoff, Russell L. The development of operations research as a science. Operations Res. **4** (1956), 265–295.

Luce, R. Duncan. Semiorders and a theory of utility discrimination. Econometrica **24** (1956), 178–191.

In the theory of utility preferences it is assumed that the indifference relation is transitive and this assumption enables the economist to introduce indifference curves. [See, e.g., von Neumann and Morgenstern, Theory of games and economic behavior, 2nd ed. Princeton, 1947, pp. 617–632; MR **9**, 50.] It has been pointed that this assumption is strictly contrary to experience, in fact a similar observation was emphasised for probability orderings by J. M. Keynes in his Treatise on probability [Macmillan, London, 1921]. The assumption is of course usually made for the sake of simplicity. In this paper axioms are suggested for intransitive indifference relations such that "an indifference interval never spans a preference interval". Very similar axioms had been independently suggested by E. Halphen [Publ. Inst. Statist. Univ. Paris **4** (1955), 41–92; MR **17**, 6] except that Halphen, like Keynes, was concerned with probabilities instead of utilities.] The axioms are shown to lead substantially to a utility theory in which for any value of utility there corresponds a change in utility that is just discernible. Cases where it is desired to discuss expected utility are considered in some detail, but the results are less satisfactory than in the theory of von Neumann and Morgenstern. *I. J. Good* (Cheltenham).

Slade, J. J., Jr. Some observations on formal models for programming. Trans. A.S.M.E. **78** (1956), 47–53.

This paper begins with a geometrical discussion of the various types of solutions arising for the problems of finding the extreme value of a function $z=f(x_1, \cdots, x_n)$ when the variables x_i are subject to certain inequality constraints (in short, the problems of linear and nonlinear programming). The special function

$$z = \sum a_i x_i \exp(-\sum \alpha_i x_i)$$

is considered in detail, and it is shown that only linear processes are involved in finding the extreme values of z subject to linear constraints. This objective function is proposed as a mathematical model for situations in which the linear factor dominates the profit function for small rates of production and a negative exponential for high rates. Several illustrative examples are included. *H. W. Kuhn* (Bryn Mawr, Pa.).

Berge, Claude. Sur une généralisation du théorème de Zermelo-von Neumann. C. R. Acad. Sci. Paris **241** (1955), 455–457.

The theorem of Zermelo-von Neumann, as extended by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 570–576; MR **12**, 515], asserts that every finite game with perfect information possesses an equilibrium point in pure strategies. Here, this result is generalized for a class of infinite games presented in the author's global extensive form [J. Math. Pures Appl. (9) **32** (1953), 129–184; MR **16**, 500]. *H. W. Kuhn* (Bryn Mawr, Pa.).

* **Gillis, Paul P.; et Huyberechts, Simone. Théorie des jeux sur le carré-unité.** Colloque sur l'analyse statistique, Bruxelles, 1954, pp. 159–176. Georges Thone, Liège; Masson & Cie, Paris, 1955.

The main portion of this paper surveys some of the known results concerning the existence of a value of a zero-sum two-person game, and the structure of the solutions to games on the unit square with special kernels. The remainder presents proposals for research on games on the unit square, in particular, structure theorems for special bounded kernels (e.g., harmonic functions), existence questions for unbounded kernels, and problems relative to the uniqueness of a solution. Examples are given to illustrate the first two areas. *H. W. Kuhn.*

Saaty, Thomas L. Approximation to the value of the objective function in linear programming by the method of partitions. Operations Res. **4** (1956), 352–353.

* **Bellman, Richard. Computational problems in the theory of dynamic programming.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 1–10. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R.I. $9.75.

This paper deals with several results concerning the functional approach to the following problem: A quantity $x>0$ is divided in two parts, y and $x-y$. From y, we obtain a return $g(y)$ and from $x-y$ a return $h(x-y)$. We apply the same process to the quantity $x'=ay+b(x-y)$, where a and b are positive constants less than one. We try to maximize the total return obtained after N stages. Several generalizations and particular cases, according to the behavior of the functions h and g, are also considered. Most of the proofs are omitted and the reader is referred to other papers of the author. One example is studied.

A. G. Azpeitia (Providence, R.I.).

★ **Ulam, S. Applications of Monte Carlo methods to tactical games.** Symposium on Monte Carlo methods, University of Florida, 1954, p. 63. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

Flood, Merrill M. The traveling-salesman problem. Operations Res. **4** (1956), 61–75.

This paper presents an apt and authoritative review of the history and connections of the so-called personnel assignment, transportation, and traveling-salesman problems. A method for solving the assignment problem [with an approach essentially the same as that of the reviewer, Naval Res. Logist. Quart. **2** (1955), 83–97; MR **17**, 759] is used in the initial preparation of a traveling-salesman problem for subsequent computations. Some techniques for improving trial solutions are given and the computational devices of Dantzig, Fulkerson, and Johnson [J. Operations Res. Soc. Amer. **2** (1954), 393–410; MR **17**, 58] are discussed. The discussion is accompanied by numerical examples, and applications of these problems as models for management problems are cited. *H. W. Kuhn* (Bryn Mawr, Pa.).

TOPOLOGICAL ALGEBRAIC STRUCTURES

Numakura, Katsumi. Theory of compact rings. II. Math. J. Okayama Univ. **5** (1956), 103–113.

[For part I, see same J. **5** (1955), 79–93; MR **17**, 642.] The author continues his study of the structure of compact (topological) rings with identity and improves many of the results of part I. In addition completely primary compact rings are studied. (A ring R is primary if it has an identity and the residue class ring of R modulo its Perlis-Jacobson radical N is simple; completely primary if it is primary and R/N is a division ring.) Only a few highlights of the paper will be described below.

If R is a compact ring with identity, then R is a complete direct sum of compact primary rings if and only if every pair of maximal open prime ideals commute if and only if every ideal of R equal to the closure of its square has an identity. If R is a completely primary compact ring and there is no left (or right) ideal of R between N and N^2, then every proper one-sided ideal of R is a power of N. This latter is equivalent to N being a principal ideal. In a compact primary ring N is a principal ideal if and only if every left, right, or two-sided ideal of R is principal.

M. Henriksen (Lafayette, Ind.).

Feldman, Jacob. Isomorphisms of finite type II rings of operators. Ann. of Math. (2) **63** (1956), 565–571.

Dye [Duke Math. J. **20** (1953), 55–69; MR **14**, 659] has shown that for finite factors not of Type I_{2n} the structure of the unitary group determines the * structure of the factor. [Also, see Sakai, Tôhoku Math. J. (2) **7** (1955), 87–95; MR **17**, 389.] The present paper extends these results to rings of operators of type II_1: Theorem 4. Let A and B be II_1 rings of operators with unitary groups $\mathbf{U}$, $\mathbf{V}$ resp. Let u be a group isomorphism from $\mathbf{U}$ onto $\mathbf{V}$. Then there is a ring, *-isomorphism g of A onto B and a multiplicative map $\tilde{u}$ from $\mathbf{U}$ to center of $\mathbf{V}$ such that $u(U)=\tilde{u}(U)g(U)$ for all $\mathbf{U}$ in $\mathbf{V}$. The proof uses the von Neumann methods and these theorems: Theorem 3. Let A be a II_1 ring of operators, and let g be an orthomorphism (an orthomorphism is a 1-1 order preserving map between lattices of projections, preserving orthogonal complements), from its lattice L of projections onto the lattice M of projections in the II_1 ring B. Then there is a ring, *-isomorphism g from A onto B agreeing with g on projections. Theorem 1. Let A be a II_1 ring of operators and C a maximal abelian self-adjoint subalgebra of A. Let A_0 be an operator in the center Z of A with $0\leq A_0\leq I$. Then there is a projection P_0 in C with trace (P_0) equal to A_0. *E. L. Griffin, Jr.* (Ann Arbor, Mich.).

Topological Groups

Isiwata, Takesi. Duality of topological groups. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1955), 82–87.

The author investigates the class of abelian groups for

which some form of duality theorem holds. Let G be a discrete abelian group and $T(G)$ the set of all topologies on G, with the natural order in T. If $t \in T(G)$ then G_t will denote G with topology t. Let G_{t*} be the discrete group of all characters of G_t. The compact-open topology on G_{t*} is denoted by $k(t)$; and $w(t) \in T(G)$ is the weakest topology on G such that each character of G_t is a character of $G_{w(t)}$. The discrete topology is denoted by d. Duality definitions are as follows: G_t has weak s-duality (or weak s-t-duality) if there is an $s \in T(G_{t*})$ such that $G_{t*s*}=G$. G_t has s-duality (or s-t-duality) if $\alpha(x)$ is continuous on $G_{t*s}\oplus G_t$ for $\alpha \in G_{t*s}$ and $x \in G_t$. The Pontrjagin duality theorem may be regarded as asserting that, for G_t locally compact, G_t has $k(t)-k(k(t))$-duality. The author proves several theorems concerning these notions. For example: If G_t is locally compact, then $G_{w(t)}$ has weak $k(t)$-duality. Also the class of locally compact groups is a proper subset of the groups with duality. *M. E. Shanks.*

Montgomery, D.; Samelson, H.; and Yang, C. T. Groups on E^n with $(n-2)$-dimensional orbits. Proc. Amer. Math. Soc. **7** (1956), 719–728.

The authors prove the following theorem: Let G be a connected compact Lie group acting differentiably on the Euclidean n-space E so that the highest dimension of any orbit is $n-2$. Then the action of G is linear in properly chosen coordinates of E.

An outline of the proof is as follows. Let S be the n-sphere obtained by adjoining a point q to E. G may be considered as acting on S leaving q fixed. Let S^* be the space of orbits of G in S and f the natural map of S onto S^*. Put $B=S-X$ where X denotes the set of points in S whose orbits have dimension $n-2$. It is first proved that S^* is a closed two-cell with the boundary $f(B)$ and that the stability group G_x of x depends continuously upon x in X. The orbits in B are then carefully studied and it is shown that there is a fixed point p in E and that $f(p)$ and $f(q)$ divide the circle $f(B)$ into two open intervals such that on the inverse (under f) of each interval, G_x depends continuously upon x. It then follows that there exist a cross-section for the orbits of G in E and, also, a homeomorphism of an open invariant neighborhood of p onto the space E which commutes with the action of G on E. Now, by a theorem of Bochner, G acts linearly on a neighborhood of the fixed point p. Hence, by the existence of the homeomorphism described above, the theorem is proved. *K. Iwasawa* (Cambridge, Mass.).

Montgomery, D.; Samelson, H.; and Yang, C. T. Exceptional orbits of highest dimension. Ann. of Math. (2) **64** (1956), 131–141.

Let G be a compact connected Lie group acting differentiably on a differentiable manifold of dimension n. Let G_x denote the group of stability of a point x on M and G_x^* the identity component of G_x. The order of G_x/G_x^* is then finite and will be denoted by $m(x)$. Let r be the highest dimension of any orbit of G on M and F the set of points on M whose orbits have dimension less than r. It is known [Montgomery, Samelson, and Zippin, Ann. of Math. (2) **63** (1956), 1–9; MR **17**, 643] that F has dimension $\leqq n-2$. Now, let k be the minimum of $m(x)$ for x in $M-F$ and E the set of all points x in $M-F$ for which $m(x)>k$. In the present paper, the authors prove that if the $(n-1)$th homology of M is 0, then $E\cup F$ is a closed set of dimension at most $n-2$. Here, the $(n-1)$th homology group of M means the $(n-1)$th Čech homology group of M or of its one-point compactification according as M is compact or not, the coefficients being the integers mod 2. It follows, in particular, that if G and M are as above, M is, except for a subset $E\cup F$ of dimension $\leqq n-2$, a fibre bundle with orbits of G as fibres.

The proof depends upon a key lemma which describes explicitly the local behavior of G acting on a neighborhood of a point on M. It is also shown by simple counter examples that neither of the assumptions on G and M, the connectivity of G and $H_{n-1}(M)=0$, can be deleted in the above theorem. *K. Iwasawa* (Cambridge, Mass.).

Wang, Hsien-Chung. Discrete subgroups of solvable Lie groups. I. Ann. of Math. (2) **64** (1956), 1–19.

Let M be a torsion free nilpotent Lie group whose component group (the factor group of M modulo the connected component of the identity) is finitely generated. A topological group L is called an S-group with kernel M if L is an extension of M by a finitely generated discrete abelian group H. If H is torsion free, L is called strongly torsion free.

The main result of the paper is then stated as follows: Given any S-group L with kernel M, there exists an imbedding j of L into a connected solvable real linear Lie group such that all the characteristic roots of any matrix in $j(M)$ are equal to 1. If L is furthermore strongly torsion free, L can be also imbedded in a connected and simply connected solvable Lie group U so that M is contained in the commutator subgroup of U.

From the above, the following interesting results are also obtained: 1) a discrete group can be imbedded in a connected and simply connected solvable Lie group if and only if it is a strongly torsion free S-group, 2) Given any S-group L, there exists a simply connected solvable linear Lie group G with only a finite number of connected components such that L is a closed subgroup of G and G/L is compact.

The most essential step in the proof is to prove that every S-group L with kernel M can be imbedded in a splittable extension L^* of a connected, simply connected nilpotent Lie group M^* by a finitely generated discrete abelian group so that M is contained in M^*. The main result then follows from a theorem of Birkhoff on representations of Lie groups, applying it to the group M^*. It is also given an example of a strongly torsion free discrete S-group L which cannot be imbedded as a closed subgroup in any connected solvable Lie group G so that G/L is compact. *K. Iwasawa* (Cambridge, Mass.).

Mostert, Paul S.; and Shields, Allen L. On a class of semigroups on E_n. Proc. Amer. Math. Soc. **7** (1956), 729–734.

A semigroup being a Hausdorff space together with a continuous associative multiplication, the paper is best described by quoting its main results:

Theorem A. Let S be the half-line $[0, \infty)$. Suppose S is a semigroup with zero at 0 and identity at 1. Then (i) if S contains no other idempotents, its multiplication is the ordinary multiplication of real numbers on $[0, \infty)$; (ii) if S contains an idempotent different from 0 and 1, then it contains a largest (in the sense of the regular order of real numbers) such idempotent e. Moreover, $e<1$, $[e, \infty)$ is a subsemigroup topologically isomorphic to $[0, \infty)$ under the usual multiplication of real numbers.

Theorem B. Let S be a semigroup with identity on Euclidean n-space, $n>1$, and B a compact connected submanifold of dimension $n-1$. If B is a subsemigroup containing the identity of S, then: (i) $n=2$ or 4 and B is a

Lie group which is S^1 if $n=2$ and S^3 if $n=4$ (where S^i denotes the i-sphere); (ii) there exists a closed subsemigroup J contained in the center of S which is topologically isomorphic to a semigroup of the type described in Theorem A; (iii) the subsemigroup J meets each orbit $xB=Bx$ of B in exactly one point, and $JB=S$; (iv) if 0 denotes the zero of J, then 0 is a zero for S, and $(J\setminus\{0\})\times B$ is topologically isomorphic to $(J\setminus\{0\})B=S\setminus\{0\}$ in the natural way. *A. D. Wallace* (New Orleans, La.).

Hewitt, Edwin. Compact monothetic semigroups. Duke Math. J. **23** (1956), 447–457.

Denote by G a Hausdorff space endowed with a continuous associative multiplication and say that G is monothetic if for some $a \in G$ the set $a, a^2, a^3, \cdots$ is dense in G. The purpose of the paper is the determination of all compact monothetic G. The author was anticipated in part by R. J. Koch [Dissertation Tulane, 1953, unpublished]. The principal result states that a compact monothetic G contains a compact monothetic group H such that (I) $G=H$ or (II) $G\setminus H=\{a, a^2, \cdots, a^s\}$, $a^{s+1}=H$, ae generates H (e the unit or H), or (III) $G\setminus H=\{a, a^2, a^3, \cdots\}$, ae generates H. The details of the algebra and topology of this situation are carefully given and it is shown how G may be "assembled" from H and the "tails" $\{a, a^2, \cdots, a^s\}$ of $\{a, a^2, a^3, \cdots\}$. The final section is devoted to the character theory and invariant integration. *A. D. Wallace* (New Orleans, La.).

See also: Ohtsuka, p. 1196; Arcidiacono, p. 1244.

Lie Groups, Lie Algebras

Sul'din, A. V. On irreducible representations of Lie algebras over fields of characteristic p. Kazan. Gos. Univ. Uč. Zap. **114**, no. 2 (1954), 167–168. (Russian)

The author shows that the degrees of the absolutely irreducible representations of a Lie algebra over a field k of characteristic $\neq 0$ are bounded. Consequently, if k is finite, the number of such representations in an extension K/k is finite if K is finite and has cardinal number equal to that of K if K is infinite. *E. R. Kolchin.*

Topological Vector Spaces

Landsberg, Max. Lineare topologische Räume, die nicht lokalkonvex sind. Math. Z. **65** (1956), 104–112.

Let E be a linear topological space and M a subset of E. If $0<r\leq 1$, M is called r-convex if for each x, y in M the arc $\{ax+by \mid a\geq 0,\ b\geq 0,\ a^r+b^r=1\}$ is in M. E is called r-convex if each neighborhood of 0 contains an r-convex neighborhood of 0 (so E 1-convex means that E is locally convex). The author gives a sufficient condition that a space not be r-convex; from it he proves that while the known spaces L_r and H_r are r-convex they are not s-convex if $r<s\leq 1$. His hypothesis on the kind of measure underlying L_r is unnecessarily restrictive; it is only necessary that the space L_r be infinite dimensional; that is, that the Boolean algebra of measurable sets mod null sets be infinite. *M. M. Day* (Seattle, Wash.).

Balanzat, Manuel. The differential in affine metric spaces. Math. Notae **9** (1949), 29–51 (1950). (Spanish)

It was pointed out by Fréchet [J. Math. Pures Appl. (9) **16** (1937), 233–250] that Hadamard's definition of the differential for ordinary functions [Scripta Univ. Atque Biblio-Hierosolymitanarum **1** (1923), no. 4] can easily be generalized to abstract vector spaces, with various topologies. The topology enters into the definition of the derivative of a vector valued function of a real variable, upon which Hadamard's definition is based.

Ky Fan [same J. (9) **21** (1942), 289–368; MR **5**, 99] has defined a differential of the Hadamard type for transformations between affine metric spaces [Fréchet, Les espaces abstraits, Gauthier-Villars, Paris, 1928, p. 144]. In such spaces there is postulated not only a metric (x, y) but also a "length" $\|\mathbf{xy}\|$ of the vector joining x and y, which, in general, has little to do with the metric, except along a straight line. In defining the derivative $f'(t)$ of a vector valued function $f(t)$, Fan uses the topology given by the length functional, and finds (not very surprisingly) that a differentiable function, in the sense of his definition, is not always continuous in the metric topology.

Balanzat remedies this deficiency by using the following definition for $f'(t)$. He requires that

$$\lim_{t\to t_0}(\omega, 0)/(t-t_0)=0,$$

where $\omega=f(t)-f(t_0)-(t-t_0)f'(t_0)$ and then shows that the Hadamard differential based upon this definition of $f'(t)$ has the usual elementary properties (uniqueness, differentiable functions are continuous, the composite function rule holds, etc.). *D. H. Hyers* (Los Angeles, Calif.).

Blackett, D. W. The near-ring of affine transformations. Proc. Amer. Math. Soc. **7** (1956), 517–519.

Associated with each linear transformation T of a vector space V and each $\beta \in V$ is an affine transformation $A=A(T, \beta)$ of V: $\alpha A=\alpha T+\beta$, $\alpha \in V$. The set N of all affine transformations of V is a near-ring with commutative addition. If $I=\{A(0, \beta)\}$, then I is an ideal of N and N/I is isomorphic to the ring of all linear transformations of V. (The author states that N/I is a simple ring; this is so if and only if V is finite dimensional.) If V is finite dimensional, N is shown to be isomorphic to a near-ring of $(n+1)\times(n+1)$ matrices. *R. E. Johnson.*

Ganapathy Iyer, V. On the space of integral functions. IV. Proc. Amer. Math. Soc. **7** (1956), 644–649.

[For parts I–III see MR **10**, 380; **12**, 108; **14**, 657.] Let Γ be the space of entire functions, under the usual topology of compact convergence. This is metrizable, with $\varrho(\alpha, \beta)=|\alpha-\beta|$, where

$$|\alpha|=\max\{|a_0|, |a_1|, \cdots, |a_n|^{1/n}, \cdots\}$$

and $\alpha(z)=a_0+a_1z+a_2z^2+\cdots$. The author first shows that a linear transformation T of Γ into itself which obeys $|T\alpha|\leq M|\alpha|$ for all $\alpha \in \Gamma$ (and in particular, the isometries) must be almost diagonal, obeying $T\delta_n=k_n\delta_n$ for $n=2, 3, \cdots$, where $\{\delta_n\}$ is the simple basis for Γ defined by $\delta_n(z)=z^n$. This depends upon a peculiar property of the non-homogeneous metric. One has $|\delta_n|=1$, but $|c\delta_n|=c^{1/n}$ for all $c>0$. Moreover, if $\alpha \in \Gamma$ and $|c\alpha|\leq Mc^{1/k}$ for all $c>0$, and a fixed integer $k\geq 2$, then α is a multiple of δ_k.

The space Γ becomes an algebra $\Gamma(C)$ under termwise multiplication of the coefficient sequences. The author shows that the general automorphism of $\Gamma(C)$ is given by $T\delta_n=\delta_{\theta(n)}$, where θ may be any permutation of the non-negative integers obeying $b\leq\theta(n)/n\leq B$ for some $b>0$, $B<\infty$. *R. C. Buck* (Madison, Wis.).

Altman, M. On linear functional equations in (B_0)-spaces. Studia Math. **15** (1956), 131–135.

This study may be regarded as an axiomatic approach to the Riesz-Schauder theory of linear operators $T=I-U$, I the identity, U completely continuous. Let X be a (B_0)-space [i.e., complete, locally convex, linear metric space; see Mazur and Orlicz, Studia Math. **10** (1948), 184–208; MR **10**, 611]. Say that a linear operator T has property R_μ if $T^{\mu+1}x=0$ implies $T^\mu x=0$; the least such μ is the order of T. The author studies linear operators T in X for which T and its adjoint T^* have finite orders μ and μ^* and also such that the ranges of the operators T^i ($i=1, 2, \cdots, \mu^*+1$) are all closed. The author derives the Riesz-Schauder pattern from this; for example, $\mu=\mu^*$; $T=H+K$ where H has a continuous inverse and the range of K is the null-space of T^μ; the Fredholm alternative holds. *M. M. Day* (Seattle, Wash.).

Al'tman, M. On the Riesz-Schauder theory of linear operator equations in spaces of type (B_0). Studia Math. **15** (1956), 136–143. (Russian)

The author uses the results of the paper reviewed above to prove the following theorem: If U is completely continuous, if $T=I-U$, and if A and B are linear operators such that $T=AB=BA$, then the Reisz-Schauder theory holds for A and for B. This has as a simple corollary the result [proved for Banach spaces by Nikol'skiĭ, C. R. (Dokl.) Acad. Sci. URSS (N.S.) **2** (1936), 315–319] that if a power of U is completely continuous, then the R-S theory holds for $I-U$. *M. M. Day* (Seattle, Wash.).

Blackett, D. W. Simple near-rings of differentiable transformations. Proc. Amer. Math. Soc. **7** (1956), 599–606.

If V is real n-space, each mapping T of V into V has the form $(x_1, \cdots, x_n)T=XT=(f^1(X), \cdots, f^n(X))$, $X \in V$. This paper is a study of the near-ring N of all mappings of V into V such that $0T=0$ and T is differentiable in a neighborhood of 0. The correspondence $T \to (f_r^s(0))$, where $f_r^s=\partial f^s/\partial x_r$, is a homomorphism of N into a ring of real $n \times n$ matrices with kernel made up of all T having a zero total differential at 0. It is shown that the ring of all linear transformations of V is a maximal simple near-ring in N. Other simple near-rings in N are classified. *R. E. Johnson* (Northampton, Mass.).

See also: Zeller, p. 1200.

Banach Spaces, Banach Algebras

Feldman, Jacob. Some connections between topological and algebraic properties in rings of operators. Duke Math. J. **23** (1956), 365–370.

1. The author constructs a dimension function in any AW^*-algebra; it takes values which are real numbers or cardinal numbers, and its domain is the space of maximal ideals of the center. 2. From this he deduces the existence of a purely infinite projection which is minimal in the sense of equivalence. 3. A theorem is proved relating the maximal number of orthogonal projections and the maximal number of equivalent orthogonal projections. 4. If the ring A of all bounded operators on a Hilbert space has a weakly dense subset of cardinal α, then A is locally α-decomposable (i.e. after a central decomposition there are most α orthogonal projections). *I. Kaplansky.*

Arens, Richard; and Singer, I. M. Generalized analytic functions. Trans. Amer. Math. Soc. **81** (1956), 379–393.

Let G denote an arbitrary locally compact abelian group and let G_+ denote a closed sub-semigroup of G whose interior is dense in G_+ and generates G. Let A_1 denote the closed subalgebra of $L_1(G)$ consisting of all functions whose support is contained in G_+. Regarding A_1 as an algebra of operators on $L_2(G)$, let A_0 denote its completion in the operator norm. Both of these algebras may be regarded as subalgebras of the algebra $C_0(\Gamma)$ of all complex-valued continuous functions on the character group Γ of G that vanish at infinity.

Both A_0 and A_1 are regarded as algebras of "generalized analytic functions", since if G is the group of integers and G_+ the sub-semigroup of non-negative integers, then Γ is the unit circle $|z|=1$ of the complex plane and A_1 (respectively A_0) becomes the subalgebra of $C_0(\Gamma)$ consisting of all functions having extension over $|z| \leqq 1$ that have absolutely convergent Taylor series on $|z| \leqq 1$ (respectively that are analytic on $|z|<1$). [See Loomis, An introduction to abstract harmonic analysis, Van Nostrand, Toronto-New York-London, 1953, p. 81; MR **14**, 883.]

In the general case, the problem of determining when an element of $C_0(\Gamma)$ represents an element of A_0 or A_1 is difficult; partial results of this kind are obtained. With the aid of some intricate analysis, and under suitable restrictions on the algebras in question, analogues of the Cauchy integral formula and other integral representations are obtained. A maximum modulus theorem is proved. Moreover, it is shown that if G_+ has sufficiently many continuous "characters", then an element of A_0 or A_1 that vanishes on a nonvoid open subset of Γ vanishes identically. Under these restrictions, A_0 and A_1 are integral domains. A by-product of this study is the existence of semi-simple Banach algebras of arbitrary large dimension. *M. Henriksen* (Lafayette, Ind.).

Arens, Richard. A Banach algebra generalization of conformal mappings of the disc. Trans. Amer. Math. Soc. **81** (1956), 501–513.

This paper continues a study of generalized analytic functions begun in the paper reviewed above. In the present paper, G is always a partially ordered discrete abelian group and G_+ is its set of nonnegative elements.

First the case of G totally ordered and archimedian is considered (in which case G is a subgroup of the group R of additive reals). Then A_0 and A_1 are identified as Banach algebras of uniformly almost periodic functions holomorphic on a half-plane [Besicovitch, Almost periodic functions, Cambridge, 1932].

In case G is the additive group I of integers, it is known that every automorphism of A_0 can be obtained from conformal mappings of the maximal ideal space Δ of A_0 and vice versa. (In this case, Δ is the disc $|z| \leqq 1$.) The author studies this relation between automorphisms of A_0 and "generalized conformal" mappings of Δ in the case when G is a weakly archimedian ordered discrete abelian group, i.e., every element of G is a difference of positive elements, and for each $v \neq 0$ in G, there is a homomorphism h of G into R such that $h(v) \neq 0$ and $h(t) \geqq 0$ for all $t \geqq 0$ in G. A generalization of the fact that when $G=I$, every conformal mapping of Δ leaves the origin fixed is obtained. Moreover, the bounded derivations of A_0 are studied.

Finally a nonarchimedian example is studied in order

to obtain natural limits on the results obtained in the archimedian case. *M. Henriksen* (Lafayette, Ind.).

San Juan, R. Generalization of a theorem of Steinhaus on linear functionals. Las Ciencias **17** (1952), 205–208. (Spanish)

The space LD is the set of all functions on the unit interval with an improper Lebesgue-Dirichlet integral. A norm can be defined on LD so that all continuous linear functionals are extensions by continuity of functionals on the space L of Lebesgue integrable functions.

D. C. Kleinecke (Albuquerque, N.M.).

Ogasawara, Tôzirô. A structure theorem for complete quasi-unitary algebras. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 79–85.

A *-algebra A which is a pre-Hilbert space under an inner product $\langle x, y\rangle$ is called quasi-unitary [Dixmier, Comm. Math. Helv. **26** (1952), 275–322; MR **14**, 660] provided it carries an automorphism $J: x \to x^J$ such that the following axioms are satisfied for all $x, y, z \in A$: (1) $\langle x, x^J\rangle \geqq 0$; (2) $\langle x, x\rangle = \langle x^*, x^*\rangle$; (3) $\langle xy, z\rangle = \langle y, x^{*J}z\rangle$; (4) the operator $U_x: y \to xy$ is continuous; (5) elements of the form $xy+(xy)^J$ are dense in A. If A is complete, axioms (4) and (5) can be replaced by the condition that $x^*x=0$ imply $x=0$. Furthermore, a complete quasi-unitary algebra can be renormed so as to become an H^*-algebra [Ambrose, Trans. Amer. Math. Soc. **57** (1945), 364–386; MR **7**, 126] with the same involution. Dixmier [loc. cit.] obtained a structure theorem for finite dimensional quasi-unitary algebras. The Dixmier result is generalized here to any complete quasi-unitary algebra. The general situation is as follows: Let M be a ring of operators (with 1) on a Hilbert space $\mathfrak{H}$ with a faithful, normal, essential pseudo-trace ψ such that $\psi(P) \geqq 1$ for every non-zero projection $P \in M$. Let H be the set of all $T \in M$ such that $\psi(TT^*) < \infty$. Then H is an H^*-algebra under the inner product $(T, T_1) = \psi(TT_1^*)$. If C is any positive definite operator in M, then H is a complete quasi-unitary algebra with automorphism $J: T \to C^{-1}TC$ and inner product $\langle T, T_1\rangle = (T, CT_1C)$. Conversely, every complete quasi-unitary algebra is of this form.

C. E. Rickart (New Haven, Conn.).

Klee, V. L., Jr. A note on topological properties of normed linear spaces. Proc. Amer. Math. Soc. **7** (1956), 673–674.

Smulian has proved a theorem [Mat. Sb. N.S. **5(47)** (1939), 317–328; MR **1**, 335] to the effect that each infinite dimensional non-reflexive space contains a decreasing sequence of non-empty bounded convex sets whose intersection is empty. The author has used this result to derive some topological properties of normed linear spaces [Trans. Amer. Math. Soc. **74** (1953), 10–43; MR **14**, 989]. In order to extend the latter results to reflexive spaces, the author proves in this note an analogue of Smulian's result, viz: Every infinite dimensional normed linear space contains a decreasing sequence of unbounded but linearly bounded closed convex sets whose intersection is empty. The emphasis here is on "linearly bounded set" which means that the set contains no linear segment of infinite length. With the help of this theorem, the author establishes four homeomorphism results for arbitrary infinite dimensional spaces E of which the following will serve as example: There is a homeomorphism of period two without fixed points of E onto E which takes the unit ball of E into itself. *E. R. Lorch* (New York, N.Y.).

McArthur, C. W. On relationships amongst certain spaces of sequences in an arbitrary Banach space. Canad. J. Math. **8** (1956), 192–197.

Verfasser betrachtet Umordnungen von Reihen in Banachräumen. $U(X)$, $B(X)$, $IS(X)$ bezeichne die Menge der Folgen $\{s(i)\}$ mit Elementen aus dem B-Raum X, bei denen jede Umordnung von $\sum s(i)$ konvergiert, jede Umordnung schwach konvergiert (nicht notwendig gegen ein Grenzelement), alle konvergenten Umordnungen dieselbe Summe liefern. [Vgl. Hadwiger, Math. Z. **47** (1941), 325–329; MR **3**, 295.] Die Teilmengen $B_w(X)$ und $B_s(X)$ von $B(X)$ erklärt Verfasser durch das Vorhandensein eines schwachen bzw. starken Grenzwertes bei $\sum s(i)$. B_w und B_s sind lineare abgeschlossene Unterräume in der in naheliegender Weise als B-Raum aufgefaßten Menge B. Stets ist

$$(*) \qquad U(X) \subset B_s(X) = IS(X) \cap B(X) \subset B_w(X) \subset B(X).$$

Ist X unendlichdimensional, so gilt $U(X) \neq IS(X)$ (für den Hilbertraum zeigte das Hadwiger, loc. cit.). Bei schwach vollständigen X dürfen in (*) alle Zeichen $\subset$ durch $=$ ersetzt werden; ist $X = c_0$ (Raum der Nullfolgen) sind alle Enthaltenseinrelationen in (*) eigentlich.

K. Zeller (Tübingen).

Leżański, T. On a representation of the resolvent. Studia Math. **15** (1956), 144–147.

To solve the equation $Ax = y_0$, for a fixed y_0 of a Banach space X with A a linear continuous transformation on X to X, the author sets $\exp A = \sum_{n=0}^{\infty} A^n/n!$ ($A^0 = I$, the identity) and $U(t, A) = \int_0^t \exp(-sA)ds$. If

$$\lim_{t\to\infty} U(t, A)y_0 = x_0$$

weakly in X, then $Ax_0 = y_0$; if $\lim_{t\to\infty} U(t, A) = U$ weakly in the space of linear continuous transformations on X to X, then $UA = AU = I$. If $F(A) = \sum_{n=1}^{\infty} A^n/n!$ and

$$V(t, A) = \int_0^t F(-sA)ds,$$

then $V + A + AV = V + A + VA$, provided

$$\lim_{t\to\infty} V(t, A) = V$$

weakly. The last two resnlts are shown to be valid in a Banach algebra, with and without unit respectively.

T. H. Hildebrandt (Ann Arbor, Mich.).

Marinescu, G. Espaces Banach de distributions. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 8, 9–13. (Romanian. Russian and French summaries)

Let Σ be the space of all real functions ϕ defined on $]-\infty, \infty[$ infinitely differentiable everywhere except possibly at zero, and such that

$$\|\phi\| = \sup\{|\phi^{(n)}(x)| : 0 \leqq n < \infty, \ -\infty < x < \infty\}$$

is finite. Then Σ is a normed linear space. In the conjugate space Σ^*, derivatives can be introduced à la Schwartz: $\langle T', \phi\rangle = -\langle T, \phi'\rangle$ for $\phi \in \Sigma$ and $T \in \Sigma^*$. An application is made to the solution of certain differential equations.

E. Hewitt (Seattle, Wash.).

Civin, Paul; and Yood, Bertram. Ideals in multiplicative semi-groups of continuous functions. Duke Math. J. **23** (1956), 325–334.

Let $C(X)$ be the Banach algebra of all real (or complex) continuous functions which vanish at infinity where X is a

locally compact Hausdorff space. The purpose of this paper is to characterize the ideals in certain multiplicative semi-groups contained in $C(X)$. Typical of the results is the following dealing with a semi-group $B \subset C(X)$ with the Šilov property: Given two disjoint closed sets $F_1, F_2 \subset X$ with F_1 compact, there exists an $f \in B$ such that $f=1$ for all $x \in F_1$ and $f=0$ for all $x \in F_2$. For such a semi-group there is a one-to-one correspondence between the closed subsets $F \subset X$ and the ideals I consisting only of divisors of zero and such that for $f_1, f_2 \in I$ there is an $e \in I$ with $ef_i=f_i$, $i=1,2$. Given two Šilov semi-groups $B_1 \subset C(X_1)$ and $B_2 \subset C(X_2)$, if B_1 is algebraically isomorphic with B_2 either as a multiplicative semi-group or under the product $f \circ g=f+g-fg$, then X_1 is homeomorphic with X_2. Results of this kind are also obtained for more general semi-groups, for instance for semi-groups $B \subset C(X)$ which are normal and dense in a Šilov semi-group; here normal is taken to mean that given an $\epsilon>0$ and two closed sets F_1, F_2 with F_1 compact, there is an $f \in B$ such that $f=0$ for all $x \in F_2$ and $|f(x)-1|<\epsilon$ for all $x \in F_1$. *R. S. Phillips.*

Domar, Yngve. Harmonic analysis based on certain commutative Banach algebras. Acta Math. **96** (1956), 1–66.

Let G be a locally compact abelian (Hausdorff) group with dual group $\hat{G}$ and character function $(x, \hat{x})$. The object of study of this monograph is a Banach algebra F of complex-valued functions on G, under pointwise operations, subject to the following 3 conditions. 1) For every neighborhood N of the identity in G, there exists a function $f_N \in F$ with support $\subset N$ and having the form $f_N(x)=\int_{\hat{G}} \overline{(x, \hat{x})} \hat{f}_N(\hat{x}) d\hat{x}$, where $\hat{f}_N$ is a continuous function in $L_1(\hat{G})$. 2) If $\hat{g}$ is a continuous function on $\hat{G}$ and $|\hat{g}(\hat{x})| \leq |\hat{f}_N(\hat{x})|$ for all $\hat{x} \in \hat{G}$, then $g(x)=\int_{\hat{G}} \overline{(x, \hat{x})} \hat{g}(\hat{x}) d\hat{x}$ is in F, and the F-norms of all such g have a common upper bound. Conditions 1) and 2) imply that for every compact $\hat{C} \subset \hat{G}$, there is an $h \in F$ such that $h(x)=\int_{\hat{G}} \overline{(x, \hat{x})} \hat{h}(\hat{x}) d\hat{x}$, $\hat{h} \in L_1(\hat{G})$, and $\hat{h}$ vanishes outside of $\hat{C}$. Condition 3) is: the class of all $h \in F$ having the property described in the last sentence is dense in F. Examples of algebras F are:

$$\{f: f(x)=\int_{\hat{G}} \overline{(x, \hat{x})} \hat{f}(\hat{x}) d\hat{x} : \hat{f} \in L_1(G)\},$$

with the L_1-norm of $\hat{f}$; $\{f: f$ is continuous on G, $f(x) \to 0$ as $x \to \infty\}$, with the uniform norm.

The first part of the paper extends certain parts of the harmonic analysis of $L_1(\hat{G})$ to all algebras F. Thus, the only multiplicative linear functionals on F are the mappings $f \to f(a)$ for $a \in G$; the weak topology for multiplicative linear functionals is the original topology for G; every closed proper ideal in F is contained in a regular maximal ideal. (It would be interesting to have a characterization of some algebras F for which every closed proper ideal is the intersection of regular maximal ideals.) In connection with the author's program for $G=$ the real line R, see J. Wermer, Ark. Mat. **2** (1954), 537–551 [MR **15**, 968].

Next, let $\hat{p}$ be a real-valued measurable function on $\hat{G}$ that in bounded on compact sets and satisfies the inequalities $\hat{p}(\hat{x}) \geq 1$ and $\hat{p}(\hat{x}_1+\hat{x}_2) \leq \hat{p}(\hat{x}_1)\hat{p}(\hat{x}_2)$. The algebra $F\{\hat{p}\}$ is defined as all functions on G of the form $f(x)=\int_{\hat{G}} \overline{(x, \hat{x})} \hat{f}(\hat{x}) d\hat{x}$, where $\hat{f} \in L_1(\hat{G})$ and $\int_{\hat{G}} |\hat{f}(\hat{x})| \hat{p}(\hat{x}) d\hat{x} < \infty$. (This algebra for $\hat{G}=R$ was introduced by A. Beurling [Nionde Skandinaviska Matematikerkongressen, Helsingfors, 1938, Mercator, Helsingfors, 1939, pp. 345–366]. Theorem: $F\{\hat{p}\}$ is an F-algebra if and only if

$$\sum_{n=1}^{\infty} n^{-2} \log [\hat{p}(n\hat{x})] < \infty$$

for every $\hat{x} \in \hat{G}$.

The last half of the paper is concerned with bounded representations of F by bounded linear operators on a normed linear space A (which can always be taken to be a Banach space). For $f \in F$, write $f \circ a$ for the value at $a \in A$ of the operator on A that corresponds to f. The spectrum is now defined not for f but for a and the representation of F, as follows. The spectrum Λ_a of $a \in A$ is the set

$$\{x: x \in G, g(x)=0 \text{ for all } g \in F \text{ such that } g \circ a=0\}.$$

This spectrum is equivalent in certain cases to the spectrum defined by Beurling [Colloq. Internat. Centre Nat. Rech. Sci., no. 15, Paris, 1949, pp. 9–29; MR 11, 429]. The author makes a very detailed study of the spectrum Δ_a, which does much to clarify the many definitions now extant. *E. Hewitt* (Seattle, Wash.).

Dieudonné, Jean. Sur la bicommutante d'une algèbre d'opérateurs. Portugal. Math. **14** (1955), 35–38.

Nonnegative decreasing functions $w_1(x)$, $w_2(x)$ are defined on a finite interval $0 \leq x \leq \gamma$ with properties: w_1, w_2, $w_1 w_2$ have finite integrals but w_1^2, w_2^2 have infinite integrals. Then w_1, w_2 are in E_2, E_1 respectively but not in E_1, E_2 respectively. Here $E_i=L_{(w_i)}{}^1$ and $L_{(w)}{}^p$ denotes the Banach space of functions with finite norm

$$\left\{\int_0^{\gamma} w(|f|^*)^p dx\right\}^{1/p}.$$

Now let E denote the Banach space of pairs $h=(h_1, h_2)$ with h_i in E_i and $\|h\|=\|h_1\|+\|h_2\|$. Let $\mathfrak{A}$ be the abelian ring of bounded linear operators on E of the form: $Tf h=(f h_1, f h_2)$; here f ranges over all bounded measurable functions. The author shows that $\mathfrak{A}'$ (the set of linear bounded operators on E which commute with every operator in $\mathfrak{A}$) consists precisely of all $T \equiv T_{u,v}$ with $Th=(uh_1, vh_2)$ with u, v bounded, measurable. Indeed, if $T(1, 0)=(u, k)$ then $T(f, 0)=(fu, fk)$ for all f in E_1 since the bounded f are dense in E_1), in particular for $f=w_2$. This forces k to be the null function and u to be bounded. Similarly $T(0, f)=(0, vf)$ for all f in E_2 for some bounded v.

As a corollary, $\mathfrak{A}''=\mathfrak{A}' \neq \mathfrak{A}$ so that the author has an example of a representation $f \to T_f$ of a Banach algebra $B(X)$ (i.e. all bounded Borel functions on a compact set X) into a ring $L(E)$ (i.e. all linear bounded operators on a Banach space E) of such a nature that if $\mathfrak{A}$ denotes the map of $B(X)$, then $\mathfrak{A}'' \neq \mathfrak{A}$. The author shows that his example can be constructed using $L_{(w)}{}^p$ with $1<p<\infty$ so that the space E is even reflexive (reviewer remarks: even uniformly convex). He refers to Dixmier [Acta Sci. Math. Szeged **12** (1950), Pars A, 213–227; MR **12**, 267] and Mackey [Commutative Banach algebras, Harvard, 1952] for the theorem that such examples are impossible if R is a Hilbert space. *I. Halperin* (Kingston, Ont.).

Widom, Harold. Embedding in algebras of type I. Duke Math. J. **23** (1956), 309–324.

The central problem of this investigation is the following: can an AW^*-algebra A with center Z be AW^*-embedded in one which is of type I and has center Z? The author proves that it is necessary and sufficient that A possess a complete set of Z-valued states which are continuous in a suitable sense. From this it is immediate that the answer is affirmative if A is finite with central

trace. It is natural to inquire further whether A, so embedded, is its own double commuter; this is true if A is finite or of type I. One suspects that both questions have unreservedly affirmative answers. The author's method in studying these question is to push far along the generalization of Hilbert space to AW^*-modules. *I. Kaplansky* (Princeton, N.J.).

Feldman, Jacob. Embedding of AW^* algebras. Duke Math. J. **23** (1956), 303–307.

The problems posed in this paper are very similar to those in the preceding paper of Widom. Given an AW^*-algebra, the author seeks an AW^*-representation as operators on a Hilbert space, paying no special attention to the center. It suffices for the algebra to have a separating set of (numerical) states which are completely additive on projections. The requirement of additivity on projections is weaker than the continuity that Widom assumed; in Lemma 3 it is shown that one can pass from the weaker to the stronger property. Theorem 1 of the paper proves that a finite algebra so embedded is weakly closed. Widom's generalization of this was inspired by the present theorem. *I. Kaplansky* (Princeton, N.J.).

See also: Bagemihl, p. 1195; Duff, p. 1213; Laugwitz, p. 1238; Gel'fand and Yaglom, p. 1261.

Hilbert Space

Povzner, A. Ya. On a class of Hilbert spaces of functions. Har'kov. Gos. Univ. Uč. Zap. **40**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **23** (1952), 27–47 (1954). (Russian)

This paper contains the full proofs of the results stated for some special Hilbert spaces of functions in Dokl. Akad. Nauk SSSR (N.S.) **68** (1949), 817–820; **74** (1950), 13–16 [MR **11**, 372; **12**, 343]. *M. M. Day* (Seattle, Wash.).

Friedman, Bernard. Operators with a closed range. Comm. Pure Appl. Math. **8** (1955), 539–550.

The principal result of the paper is as follows. Let V be a linear operator with dense domain in a Hilbert space $\mathfrak{H}$, and suppose that the ranges of V and its adjoint V^* are both closed. Then the linear operator L with the same domain, defined by

$$Lx=Vx+a\langle x, b\rangle$$

also has a closed range. Using the fact that a compact linear operator C is a uniform limit of operators of finite rank, the author deduces that $C+\lambda I$ has a closed range for every $\lambda\neq 0$, and hence obtains a new proof of the Fredholm alternative theorem for compact operators.

[Reviewer's note. (i) The author writes $\langle b, x\rangle$ instead of $\langle x, b\rangle$, suggesting that he is not using the inner product notation with the convention usual in complex Hilbert space. (ii) An extension of the arguments to Banach spaces would not be quite as easy as the author suggests in a footnote.] *F. Smithies* (Cambridge, England).

See also: Šapiro, p. 1181; Harazov, p. 1216; Ogasawara, p. 1227.

TOPOLOGY

Whyburn, G. T. Topological analysis. Bull. Amer. Math. Soc. **62** (1956), 204–218.

Cet intéressant mémoire est le texte de la conférence présidentielle faite par l'auteur à la réunion annuelle de l'A.M.S. (Dec 55). Remarquable exposition des méthodes utilisées et des résultats obtenus dans la mise en evidence du caractère spécifiquement topologique de nombreux théorèmes d'analyse, surtout dans le domaine des fonctions analytiques d'une variable complexe. Référence est faite aux travaux fondamentaux de Stoilow, Eilenberg, Kuratowski, Morse, Heins etc. Deux autres exposés antérieurs de l'auteur ont déjà été consacrés à cette question [Mem. Amer. Math. Soc. no. 1 (1950); Lectures on fonctions of a complex variable, Univ. of Michigan Press, 1955, pp. 1–14; MR **13**, 764; **16**, 1140].

La première partie de l'exposé actuel (§ 2–11) montre quelles sont les notions topologiques qui doivent jouer un rôle essentiel en analyse: applications ouvertes et fermées; caractérisation (problème non résolu) des espaces tels que tout sous ensemble homéomorphe à un ouvert soit ouvert; applications quasi-compactes (§ 4) qui, avec la notion de semi-continuité d'une décomposition permettent de caractériser les applications ouvertes; applications quasi-ouvertes (§ 6) qui fournissent une condition très large pour un résultat fondamental concernant les applications plan sur plan;car actérisations topologiques des fonctions différentiables [cf. Eggleston et Ursell, J. London Math. Soc. **27** (1952), 260–271; MR **13**, 926]; invariance des variétés à deux dimensions par les applications ouvertes légères (§ 9); théorème du module maximum et conséquences (§ 10); conservation des propriétés de légèreté et ouverture par passage à la limite (uniforme).

La deuxième partie indique des résultats nouveaux concernant la conservation de la dimension et la non densité (K non dense dans X si K ne contient aucun ouvert de X) des compacts par certaines applications ouverts (§ 12). Les résultats sont précisés davantage pour des applications quasi-ouvertes de variétés à deux dimensions dans des variétés à deux dimensions avec des conditions sur la structure de l'image inverse d'un point, conditions qui jouent un grand rôle dans ces questions. Des applications remarquables sont indiquées pour les fonctions continues différentiables et l'auteur obtient une élégante caractérisation topologique des pôles de $f(z)$ sans recours au développement en série; les deux derniers théorèmes se rapportent au comportement de $f(z)$ à la frontière du domaine où elle est définie. *L. Fourès.*

Banaschewski, Bernhard. Local connectedness of extension spaces. Canad. J. Math. **8** (1956), 395–398.

Let E be a dense subset of the topological space E^*. For a point u in E^* but not in E let $\mathfrak{S}(u)$ (called trace filter) be the class of neighborhoods of u intersected with E. From the study of trace filters these results are obtained: if E is not locally connected, or is locally compact and denumerable at infinity, then the Čech, Alexandroff (α'), and Katětov extensions are not locally connected. *R. Arens* (Los Angeles, Calif.).

Fujiwara, Kaichirô. Une note sur l'espace produit d'un espace topologique par lui-même. Math. J. Okayama Univ. **5** (1956), 121–125.

The author considers the following two properties of a topological space E. (I) If A and B are closed subsets of E and W is an open subset of $E\times E$ such that $A\times B\subset W$, then there exist open subsets U and V of E such that $A\times B\subset U\times V\subset W$. (II) If $a\in E$ and W is an open subset of $E\times E$ such that $\{a\}\times E\subset W$, then there is a neighborhood

U of a such that $U \times E \subset W$. He shows that a Hausdorff space E satisfies (I) if and only if it is normal and satisfies (II). He shows next that any topological space of compact character in the sense of T. Inagaki [same J. **2** (1953), 149–184; MR **14**, 1001] satisfies (II), and if every point of E has a totally ordered base of neighborhoods, then (II) implies that E is of compact character. It follows that if E satisfies the first axiom of countability then E satisfies (II) if and only if it is either countably compact or discrete. Finally it is shown that if every binary open covering of a uniform space E has the Lebesgue property in the sense of K. Iséki [Proc. Japan Acad. **31** (1955), 220–221, 270–271, 441–442, 524–525, 618–619; MR **17**, 389], then E satisfies (I). *M. Henriksen.*

Behrend, F. A. Note on the compactification of separated uniform spaces. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 269–270.

The author gives a simple method of obtaining for any uniform space S a uniform structure which is totally bounded and compatible with the topology of S. This result is useful since the completion of a separated uniform space is compact if its uniform structure is totally bounded. *D. W. Hall* (Endicott, N.Y.).

Ellis, D. On the topolattice and permutation group of an infinite set. II. Proc. Cambridge Philos. Soc. **50** (1954), 485–487.

Let S be an infinite set and let Λ be the lattice of all T_1-topologies on S, studied by the writer and by R. W. Bagley in another communication [Math. Japon. **3** (1954), 63–70; MR **16**, 1041]. It is proved here that the automorphism group of Λ is isomorphic to the permutation group of S. *E. Hewitt* (Seattle, Wash.).

Tominaga, Akira; and Tanaka, Tadashi. Convexification of locally connected generalized continua. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 301–306.

The author defines a generalized continuum to be a locally compact, connected, separable metric space, and proves that in each locally connected generalized continuum a topology-preserving convex metric may be defined. This weakens the hypotheses of the convexification problem proposed by Menger [Math. Ann. **100** (1928), 75–163] by replacing "compact" by "locally compact". The proof leans heavily upon the method devised by Bing [Bull. Amer. Math. Soc. **55** (1949), 812–819; MR **11**, 194] to solve the convexification problem for Peano continua. *L. M. Blumenthal* (Columbia, Mo.).

Kuratowski, K. On a characterisation of connected domains in locally connected spaces. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 211–214.

Let X be a locally connected space. If D is a regular open set (i.e. $D=\operatorname{Int}\bar{D}$) in X, then the boundary of D can be expressed as the closure of a union $\bigcup_\nu I_\nu$, where each I_ν is an irreducible cutting of X between two points. Conversely, every connected open set D in X with this property is a regular open set. This is the main result of the paper. As applications of this theorem, the author proves: (1) Let D, E be two open connected subsets of the n-sphere S^n whose boundaries are homeomorphic. If D is a regular open set, so is E. (2) If C is a locally connected continuum in S^2 and if S^2-C is a regular open set, then the boundary of C is the closure of a countable union of simple closed curves. *Ky Fan* (Notre Dame, Ind.).

Cassina, Ugo. La nozione di arco di linea nella teoria degli insiemi. Rend. Sem. Mat. Fis. Milano **25** (1953–54), 73–92 (1955).

In this largely expository article the author reviews Lennes' characterization of an arc and then describes his papers [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)** (1951), 75–88, 89–101, 102–110; MR **14**, 1002] in which he has characterized arcs without using the axiom of choice. *M. E. Shanks* (Los Angeles, Calif.).

Saeki, Takuya. On pseudorecurrence in topological dynamical systems. Tôhoku Math. J. (2) **7** (1955), 171–176.

Let X be a topological space and T a topological semigroup acting as a transformation semigroup on X. It is assumed that T has a countable base of subsemigroups α. A point $x \in X$ is said to be pseudorecurrent under T provided $x \in U_x A$ for every neighborhood U_x of x and every $A \in \alpha$. $x \in X$ is said to be regionally recurrent under T provided $U_x \cap U_x A \neq 0$ for every neighborhood U_x of x and every $A \in \alpha$. If a point is not regionally recurrent, it is said to be wandering. It is shown that if X satisfies the second axiom of countability then the sum of the pseudorecurrent points under T and the wandering points under T form a residual set in X. In case X is a complete separable metric space, pseudorecurrence is characterized by incompressibility properties. [For related theorems see W. H. Gottschalk and G. A. Hedlund, Trans. Amer. Math. Soc. **65** (1949), 348–359; MR **10**, 718; and C. W. Williams, Proc. Amer. Math. Soc. **2** (1951), 798–806; MR **13**, 256.] *W. R. Utz* (Columbia, Mo.).

Barbašin, E. A. On the behavior of points under homeomorphic transformations of a space. (Generalization of theorems of Birkhoff.) Ural. Politehn. Inst. Trudy. **51** (1954), 4–11. (Russian)

Detailed proofs of results announced in C. R. (Dokl.) Acad. Sci. URSS (N.S.) **51** (1946), 3–5; MR **8**, 34.

Anderson, R. D. Open mappings of compact continua. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 347–349.

This is a summary of recent important work of the author concerning monotone open maps; proofs will appear later. M will denote a triangulable n-manifold, with or without boundary. (I) Suppose g is a monotone open map of a compact subset K of M with $\dim K \leq n-2$ onto a space Y. Then there is a monotone open extension f of g such that $f(M-K)$ is disjoint from Y and is homeomorphic to $M-K$. (II) If $n \geq 3$, then every monotone image Y of M is also a monotone open image of M. (III) If M is either an n-cell or an n-sphere, $n \geq 3$, then every k-cell and k-sphere, $k \geq 2$, is a monotone image of M, and hence also a monotone open image of M. (IV) Let S be a continuum in M, $n>2$, let $k<n$ be a positive integer, and let $\epsilon>0$. There exists a continuum R in M and an open map f of R onto S such that (1) for each nondegenerate continuum K in R, $K=f^{-1}f(K)$, (2) $\dim R=k$, (3) for each $s \in S$, $f^{-1}(s)$ contains a k-cell, and (4) for $s \in S$ the Hausdorff distance between s and $f^{-1}(s)$ is $<\varepsilon$. (V) If M is a 2-cell or a 2-sphere, there is a monotone open map of M onto M with each $f^{-1}(x)$ a pseudo-arc. (VI) If X is the universal curve and Y is a locally connected continuum, then there exists a monotone open map of X onto Y such that for each $y \in Y$, $f^{-1}(y)$ is homeomorphic to X. *E. E. Floyd* (Charlottesville, Va.).

Ford, G. W.; and Uhlenbeck, G. E. Combinatorial problems in the theory of graphs. I. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 122–128.

Ford, G. W.; Norman, R. Z.; and Uhlenbeck, G. E. Combinatorial problems in the theory of grahps. II. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 203–208.

A short connected account of work concerned with the counting of various types of linear graphs; the topics considered arose in connexion with problems in statistical mechanics.

A connected graph is said to be divided by its articulation points (if any) into stars, and the graphs are thought of as the union of stars of varying structure.

The following topics are treated in each of the two parts: 1. The number of rooted graphs, i.e. graphs in which one point is particularly preferred. 2. The number of graphs. 3. The number of connected graphs and of stars with a given number of points and lines. In Part I all points are considered as labeled, in Part II the points are considered as indistinguishable. *G. A. Dirac.*

Nordhaus, E. A.; and Gaddum, J. W. On complementary graphs. Amer. Math. Monthly **63** (1956), 175–177.

Two graphs G and G' having the same notes are called complementary if each edge determined by the nodes is in G or in G' but not in both. It is shown that if G and G' are complementary graphs on n nodes whose chromatic numbers are k and k' respectively then

$$4n \leqq 4kk' \leqq (k+k')^2 \leqq (n+1)^2.$$

If G contains no edges then $k=1$ and $k'=n$. Examples are given for infinitely many values of n in which $k+k'=2\sqrt{n}$ and others in which $kk'=\frac{1}{4}(n+1)^2$. *G. A. Dirac.*

Kruskal, Joseph B., Jr. On the shortest spanning subtree of a graph and the traveling salesman problem. Proc. Amer. Math. Soc. **7** (1956), 48–50.

This paper proposes three simple constructions to solve the following problem: Find a spanning subtree of minimum length in a (finite) connected graph in which the edges have been assigned distinct positive real lengths. One of these constructions is used to prove that the shortest spanning tree is unique. The connection with the traveling-salesman problem seems slight. *H. W. Kuhn.*

Harary, Frank. Note on the Pólya and Otter formulas for enumerating trees. Michigan Math. J. **3** (1955–56), 109–112.

t_n' denotes the number of non-isomorphic trees with $n+1$ vertices whose centre of mass consists of one vertex and t_n'' denotes the number of non-isomorphic trees with $n+1$ vertices whose centre of mass consists of two vertices. Then the total number of non-isomorphic trees with $n+1$ vertices, denoted by t_n, is equal to $t_n'+t_n''$. T_n denotes the number of non-isomorphic rooted trees with $n+1$ vertices. Let

$$T(x)=\sum_{n=0}^{\infty} T_n x^n \text{ and } t(x)=\sum_{n=0}^{\infty} t_n x^n.$$

Pólya [Acta. Math. **68** (1937), 145–254] proved that

$$T(x)=\exp \sum_{r=1}^{\infty} \frac{x^r}{r} T(x^r) \tag{1}$$

from which $T_0, T_1, T_2, \cdots$ can be calculated recursively. He also proved (loc. cit. 203, 207) that

$$t_{2m}''=0;\ t_{2m+1}''=\binom{T_m+1}{2} \tag{2}$$

and t_n' = the coefficient of x^n in the power series expansion of

$$\prod_{s=0}^{[\frac{1}{2}n]-1} (1-x^{s+1})^{-T_s}. \tag{3}$$

More recently Otter [Ann. of Math. (2) **49** (1948), 583–599; MR **10**, 53] established the functional equation

$$t(x)=T(x)-\frac{x}{2}[T^2(x)-T(x^2)] \tag{4}$$

using a new property of trees discovered by himself.

In the present paper the author derives (4) directly from (1), (2) and (3) by algebraic manipulation, without using any properties of trees. *G. A. Dirac* (Vienna).

Plunkett, Robert L. A topological proof of a theorem of complex analysis. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 425–426.

A topological proof is given for the fact that a mapping generated by an analytic function f of a complex variable can be a local homeomorphism at a point z_0 of the complex plane if $f'(z_0)\neq 0$. Thus it is now possible to identify by topological methods the set of zeros of the derivative of such a function with the set of points where the mapping fails to be locally topological. *G. T. Whyburn.*

See also: Kogan, p. 1176; Trjitzinsky, p. 1247; Okada, p. 1258.

Algebraic Topology

Inaba, Eizi. On cohomology groups in a field, which is complete with respect to a discrete valuation. Nat. Sci. Rep. Ochanomizu Univ. **6** (1955), 25–29.

This paper deals with cohomology groups of infinite Galois groups, defined according to the following general scheme: let G be a group, M a G-module. For every subgroup U of G, let M_U denote the U-fixed part of M. If U and V are two normal subgroups of finite index in G, and $U \supset V$, one has the natural homomorphisms

$$H^n(G/U, M_U) \to H^n(G/V, M_V)$$

between the usual positive dimensional ($n>0$) cohomology groups. $H^n(G, M)$ is then defined as the direct limit belonging to this system of homomorphisms; this is not the usual cohomology group for G in M.

Now let K be a field which is complete for a discrete valuation and whose residue field $\mathfrak{K}$ is perfect. Let L be the maximal separable algebraic extension of K, $\mathfrak{L}$ the residue field of L, G the Galois group of L/K, $\mathfrak{G}$ the Galois group of $\mathfrak{L}/\mathfrak{K}$. For any field F, let F^* stand for the multiplicative group of F. Finally, let Z stand for the additive group of the rational integers, regarded as a $\mathfrak{G}$-module with trivial operators. The author shows first (by known methods) that $H^n(G, L^*)$ is isomorphic with the direct product $H^n(\mathfrak{G}, Z)\times H^n(\mathfrak{G}, \mathfrak{L}^*)$, for all $n>0$.

In the case where the generalized local class field theory holds over K, $H^n(\mathfrak{G}, \mathfrak{L}^*)$ is trivial, and the author determines the $H^n(G, L^*)$ completely. He finds that these groups are trivial, except for $n=2$, and $H^2(G, L^*)$ is isomorphic with R/Z, where R denotes the additive group of the rational numbers. The additional assumptions on K are actually the following: (1) every finite dimensional division algebra whose center is a finite algebraic extension of $\mathfrak{K}$ is commutative; (2) every finite algebraic extension of $\mathfrak{K}$ is cyclic; (3) for every positive integer n, there exists one and only one cyclic extension of $\mathfrak{K}$ of degree n. *G. P. Hochschild* (Berkeley, Calif.).

Homma, Tatsuo. On some properties of sphere sections of the sets in E^n. Yokohama Math. J. **2** (1954), 145–149 (1955).

Several theorems are proved concerning intersections of spheres in euclidean n-space E^n with subsets of E^n. For example, if $D \subset E^n$ is a domain with finite 1-dimensional Betti number, then it is proved that there is a $\delta > 0$ such that if S is an $(n-1)$-sphere of radius $< \delta$, then every component of $S \cap D$ divides D into exactly two domains. Also, if $M \subset E^n$ is a locally connected continuum and if 0 is a point of M then $\lim_{t \to r} M \cap S_t(0)$ ($S_t(0)$ the sphere of radius t centered at 0) exists and is equal to $M \cap S_r(0)$ for all but a countable number of r.

E. E. Floyd (Charlottesville, Va.).

Copeland, Arthur H., Jr. The Pontrjagin ring for certain loop spaces. Proc. Amer. Math. Soc. **7** (1956), 528–534.

Let X be a 1-connected space of Lusternick-Schnirelman strong category 2, and let Ω be its loop-space. The author shows that under these conditions the Pontrjagin ring, $H^*(\Omega; A)$ (relative to a principal ideal ring A) is completely determined by the homology of X relative to A. The algebraic step from $G = H(X; A)$ to $T^*(G) = H^*(\Omega; A)$ is as follows. Let C be any free chain complex over A, with $H(C) = G$. In the full tensor ring $T(C) = \sum C^{(n)}$ ($C^{(0)} = A$; $C^{(n)} = C \otimes C^{(n-1)} n > 0$) define a differential operator D as the linear extension of: $Da = 0$ for $a \in C^{(0)}$; D agrees with the differential operator of C on $C^{(1)}$; by induction $D(c \otimes c') = Dc \otimes c' + (-1)^{\dim c} c \otimes Dc'$ for $c \in C^{(1)}$, $c' \in C^{(n-1)}$. By definition $T^*(G)$, the torsion tensor ring of G, is the homology ring $H(T(C))$. *R. Bott* (Princeton, N.J.).

★ **Boltyanskiĭ, V. G. Gomotopičeskaya teoriya nepreryvnyh otobraženiĭ i vektornyh poleĭ. [Homotopy theory of continuous mappings and of vector fields.]** Trudy Mat. Inst. Steklov. no. 47. Izdat. Akad. Nauk SSSR, Moscow, 1955. 199 pp. (Russian)

Despite the suggestion of strong specialization conveyed by its title, this memoir contains a rather systematic introduction to combinational topology. In fact, the first part of the work, which covers nearly half the book, is subtitled 'homology theory' and deals with the homology and cohomology of simplicial complexes and cell-complexes with a special section on manifolds. The cell-complexes treated here are essentially those of J. H. C. Whitehead but the bibliography refers the readers to an early paper [On C^1-complexes] and omits all mention of the series of post-war papers on combinational homotopy. Steenrod squares are defined in the final section of the first part with a view to their subsequent application.

The second part of the book is entitled 'homotopy theory and its applications'; here the emphasis is far more strongly on the application to obstruction theory. The opening sections consist of a brief treatment of homotopy groups and the basic notions of obstruction theory. The Hopf classification theorem (for maps $K^n \to S^n$) is given together with Whitney's generalization, and there follows a section on relations between homotopy and homology. Then vector fields on manifolds are defined with special reference to Stiefel manifolds. In the next section the groups $\pi_{n+1}(S^n)$ are computed by the Pontrjagin method; the reviewer was unable to find any reference to Freudenthal's fundamental work. The last two sections discuss the problem of the second obstruction, first to the extension of a map and second to the construction of a vector field.

This book does not represent itself as a foundation course in homology and homotopy; and, obviously, certain topics have been excluded (singular homology, suspension, general theory of fibre-spaces) because they were not relevant to the particular problems of algebraic topology which the author wished to discuss. None the less, the book is probably the most comprehensive of its kind to be published and it is to be hoped that, in translation, it may reach a wider public.

[Reviewer's note: P. M. Cohn has pointed out that the statement of Sperner's Lemma on p. 18 is wrong. The usual hypothesis (in the author's notations, that, for all s, $1 \leqq s \leqq r$, the face $[b_{i_0}, b_{i_1}, \cdots, b_{i_{s-1}}]$ of the simplex T^r is contained in $F_{i_0} \cup \cdots \cup F_{i_{s-1}}$) is replaced by the weaker hypothesis that this holds for $s = r$ only, and the author's proof contains the erroneous assertion that the weaker hypothesis implies the stronger.] *P. J. Hilton.*

Dyer, Eldon. A fixed point theorem. Proc. Amer. Math. Soc. **7** (1956), 662–672.

A compact continuum M is 'chainable' or 'snakelike' if it has arbitrarily fine open coverings whose nerves are simplicial arcs. O. H. Hamilton proved that M has the fixed point property [same Proc. **2** (1951), 173–174; MR **12**, 627]. The author proves that the finite product C of chainable M's also has the fixed point property, first directly and secondly by proving that C is a zero-cyclic quasi-complex. Added in proof are two remarks: (1) both proofs can be generalized to compact Hausdorff spaces satisfying the above condition on coverings, (2) the second proof can be simplified by using the result of T. R. Brahana that the product of two quasi-complexes is a quasi-complex [Bull. Amer. Math. Soc. **62** (1956), 33].

S. Stein (Davis, Calif.).

Cohen, Haskell. Fixed points in products of ordered spaces. Proc. Amer. Math. Soc. **7** (1956), 703–706.

The author shows that the direct product of two compact connected ordered spaces has the fixed point property. This is a special case of a simultaneously published result of the reviewer [see the paper reviewed above]. *E. Dyer* (Princeton, N.J.).

Capel, C. E.; and Strother, W. L. A space of subsets having the fixed point property. Proc. Amer. Math. Soc. **7** (1956), 707–708.

Let $S(X)$ denote the space of non-null closed subsets of X. The main theorem proved is that if X is a CAR* (=retract of a Tychonoff cube), then $S(X)$ has the fixed point property. This is proved by first considering the case where X is a Tychonoff cube and then using the theorem that if X is a retract of Y (X, Y each compact Hausdorff) then $S(X)$ is a retract of $S(Y)$. *S. Stein.*

Ulčar, Jože. Über die nichtinzidenten Seiten einiger Polytope. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **8** (1955), 3–25. (Serbo-Croatian. German summary)

Für ein Komplex K sei $[K]$ die Menge aller Seiten von K; die Menge $[K]$ sei durch die Relation $\subseteq$ geordnet. Es handelt sich dann um die Bestimmung der grössten Antiketten von $[K]$ d.h. von grössten Teilsystemen A von $[K]$, die von nichtinzidenten Elementen bestehen. Der Ref. hat das Problem für Simplexe gelöst [Bull. Soc. Math. Phys. Serbie **5** (1953), no. 3–4, 23–31; MR **16**, 227]; der Verf. beweist nun diese Resultate für einige Polytope.

Insbesondere, für jedes reguläre Polytop K_n besteht A, wenn dazu $|A|$ maximal sein soll, aus Seiten von K_n, die alle von derselben Dimension sind; wenn $n\equiv 1 \pmod 6$ oder $n\equiv 3 \pmod 6$, existiert nur eine solche maximale Antikette; wenn $n\equiv 2 \pmod 6$, existieren zwei solche Antiketten; in keinem Falle kann man mehr als 2 solche maximale umfangsreichsten Antiketten M haben. Insbesondere im Raume R_4 hat man 6 reguläre Polytope Z_n: Z_5, Z_{16}, Z_{24}, Z_{120}, Z_{600} (in Z_n bedeutet n die Anzahl der begrenzenden Grenzräume von Z_n). Im Falle Z_{120} (Z_{600}) besteht M aus aller 2-(1-) dimensionalen Seiten. Für $n>4$ enthält R_n drei reguläre Polytope: A_n (der Simplex), B_n (das Masspolytop) und C_n (reziprok verwandt zu B_n). Für B_n, C_n existiert eine oder 2 Mengen M, je nachdem ob $n\neq 2 \pmod 3$ oder $n\equiv 2 \pmod 3$ ist. Für jedes simpliziale Polytop S im R_n mit $n+2$ Eckpunkten, besteht M aus allen $[\frac{1}{2}n]$-dim. Seiten von S. Wenn ein Polytop im R_n gerade $n+2$ Grenzräume R_{n-1} hat, besteht dann M aus aller $[\frac{1}{2}(n-1)]$-dim. Seiten desselben Polytops.

G. Kurepa (Zagreb).

See also: Rosenberg and Zelinsky, p. 1181.

GEOMETRY

Dingler, Hugo. Geometrie und Wirklichkeit. III. Dialectica **10** (1956), 80–93.

Stubban, John Olav. Axiomatic foundation of the euclidean geometry. Nordisk Mat. Tidskr. **4** (1956), 76–84, 119–120. (Norwegian. English summary)

★ **Weitzenböck, Roland W. Der vierdimensionale Raum.** Wissenschaft und Kultur, Band **10**. Birkhäuser Verlag, Basel und Stuttgart, 1956. 223 pp. DM 19.55.

An enlarged and revised version of the well-known book published by Vieweg [Braunschweig, 1929]. The present volume, written in the same popular but accurate style, devotes one chapter to the purely mathematical aspects of the subject, one to the theory of relativity, one to the fourth dimension in a wide variety of fields, physics, chemistry etc., and a final chapter to the imaginative literature. Numerous footnotes and a good bibliography give the book considerable historical interest.

S. H. Gould (Providence, R.I.).

ApSimon, H. G. Almost regular polyhedra. Math. Gaz. **40** (1956), 81–85.

A polyhedron is said to be "regular" if it admits two particular symmetry operations: one which cyclically permutes the edges of a face, and one which cyclically permutes the edges at a vertex belonging to that face. In three dimensions, an equivalent description is as follows: a polyhedron is regular if it has equal regular faces and equal regular vertex figures. S. Melmore has drawn attention to a polyhedron which is "almost regular" in that it has equal regular faces (namely thirty squares) and regular vertex figures (triangles and skew hexagons). The author's chief purpose is to show that, in seven or more dimensions, a polyhedron may have equal regular faces and equal regular vertex figures and yet not be regular. The following example is slightly simpler than his, but requires more than seven dimensions. Consider a parallelogram with angle 60° and sides m, n (integers greater than 2), divided in the obvious manner into mn rhombs of side 1, and so ultimately into $2mn$ equilateral triangles. Identifying opposite sides of the parallelogram, we obtain, on a torus, a topological map of type $\{3, 6\}$ which is not regular as it does not admit a symmetry operation permuting the sides of a face. This map can be realised metrically as a skew polyhedron having the same mn vertices as a regular simplex α_{mn-1} in Euclidean $(mn-1)$-space.

H. S. M. Coxeter (Toronto, Ont.).

Naumann, Herbert. Beliebige konvexe Polytope als Schnitte und Projektionen höherdimensionaler Würfel, Simplizes und Masspolytope. Math. Z. **65** (1956), 91–103.

In Euclidean n-space ($n>4$) there are just three regular polytopes: the regular simplex α_n, the cross polytope β_n, and the measure polytope γ_n [see, e.g., the reviewer's "Regular polytopes", Methuen, London, 1948; MR **10**, 261]. They occur also when $n\leqq 4$; e.g., α_3 is the tetrahedron, β_3 the octahedron, γ_3 the cube. It is unfortunate that the author uses for β_n the name "Masspolytop" which was used for γ_n by P. H. Schoute [Mehrdimensionale Geometrie, **2**, Goschen, Leipzig, 1905, p. 243]. The two reciprocal polytopes β_n, γ_n are given by

$$\sum |x_i|=R,\ \max |x_i|=r$$

in ordinary Cartesian coordinates, and by

$$\max |u_i|=R^{-1},\ \sum |u_i|=r^{-1}$$

in tangential coordinates. The author regards them as special cases of a "Hauptpolytop" which is any polytope admitting as symmetry operations the reversal of sign of one coordinate and the cyclic permutation of the n coordinates. He proves that any convex "Hauptpolytop", having $2p$ vertices and $2m$ cells, may be regarded as an orthogonal projection of β_p and as a central section of γ_m. [For the case when $n=3$, see Coxeter, op. cit., p. 254.] Since γ_m is a central section of α_{2m-1} [Schoute, op. cit. p. 253], it follows that the "Hauptpolytop" is an orthogonal projection of α_{2p-1} and a central section of α_{2m-1}. He proves further, that any convex polytope, having p vertices and m cells, may be regarded as a parallel projection of β_{kp} and as a section of γ_{km}, where $k=2^n(n+1)$.

H. S. M. Coxeter (Toronto, Ont.).

Gordevskiĭ, D. Z. The Möbius configuration in an odd-dimensional projective space. Har'kov. Gos. Univ. Uč. Zap. **29**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **21** (1949), 77–78. (Russian)

The configuration of Möbius may be regarded as consisting of two incident sets of four skew lines [see, e.g., Coxeter, Bull. Amer. Math. Soc. **56** (1950), 413–455, p. 445; MR **12**, 350]. The author generalizes this as follows. In projective $(2k+1)$-space, let α_1, α_2, α_3 be three skew k-spaces of general position. Through any point A_1 on α_1 there is a line $A_1\alpha_2\cdot A_1\alpha_3$ which is a transversal of the three k-spaces. Let p_1, p_2, $\cdots$, p_{k+2} be $k+2$ such transversals. Then any line that meets the three α's meets any k-space that meets the $k+2$ p's, and the α's belong to a singly infinite family of such k-spaces.

H. S. M. Coxeter (Toronto, Ont.).

Vincensini, Paul. Sur certaines transformations projectives de l'espace règlé, et leurs images dans l'espace euclidien à quatre dimensions. Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 311–323.

In a previous paper [same Rend. **13** (1954), 185–203 MR **16**, 955] the author defines a correspondence C be-

tween the lines with Plückerian coordinates p_{ij} in a three-dimensional Euclidean space E_3 and the points with coordinates $x_1, \cdots, x_4$ in a four-dimensional Euclidean space E_4, of which E_3 is the hyperplane $x_4=0$. In this paper under the correspondence C the image transformations in E_4 of projectivities in E_3 transforming a linear complex into a linear complex are discussed, and then it is deduced that if I is an isotropic congruence in E_4 having a surface for its trace in E_3 and if S is the transformed surface of I by the correspondence C, then S is the most general surface of Wilczynski associated with the two linear complexes corresponding to the congruence I. The author also studies the image transformations in E_4 of the four types of involutory projectivities in E_3 formed by the biaxial and central involutions and the polarities with respect to a linear complex and a quadric respectively. *C. C. Hsiung* (Bethlehem, Pa.).

Checcucci, Vittorio. Sulla riduzione a forma canonica delle equazioni di una omografia. Ann. Scuola Norm. Sup. Pisa (3) **9** (1955), 201–206 (1956).

Verf. gibt eine einfache geometrische Herleitung der Jordanschen Normalform für eine projektive Abbildung W eines complexen n-dimensionalen projektiven Raumes S_n auf sich, indem er die Zerlegbarkeit von S_n in endlichviele elementfremde Unterräume benutzt, deren jeder bei W in sich transformiert wird, wobei genau ein Punkt fest bleibt. Ferner gewinnt Verf. zugleich die bekannten Kriterien von Weierstrass und Segre dafür, dass zwei Homographien projektiv identisch sind. *R. Moufang.*

Hély, Jean. Une généralisation du théorème de Pascal: le théorème des trois courbes. Rev. Sci. **92** (1954), 32–33.

In Pascal's theorem the conic and the two triads of opposite sides are replaced by three algebraic plane curves having p points in common and which are, respectively, of orders l, m, n, with $l \leqq m \leqq n$.

Two general cases are considered: 1) $l+m-n>0$ and $p \geqq lm-(l+m-n-1)(l+m-n-2)/2$, and 2, $l+m-n \leqq 0$ and $p=lm$.

Under further restrictions several propositions are obtained of which the following (case 1b) may serve as an example: If $l<n$, the $m(n-l)$ points common to the curves of orders m and n which do not belong to the curve of order l lie on a determined curve of order $n-l<m$. Only outline of proofs are given. The paper concludes with the following application. Given an octagon inscribed in a conic, if the two points of intersection are constructed of each side with the two sides adjacent to its opposite side, the eight points obtained lie on a conic. *N. A. Court.*

Handest, Frans. Constructions in hyperbolic geometry. Canad. J. Math. **8** (1956), 389–394.

The author proves that, in the hyperbolic plane, any construction that can be performed by means of ruler and compasses can be performed by means of a "parallel-ruler", which will draw, through a given point, a line parallel to a given ray. It was proved by Hjelmslev and Forder [Math. Gazette **37**, 203 (1953)] that any construction that can be performed by means of ruler and compasses can still be performed if the compasses have a fixed adjustment, so that they will only draw circles of one particular radius. The author proves that these compasses may be replaced by a "hypercompass" which will draw the locus of points at one particular distance from any given line. [For constructions with an adjustable hypercompass but no ruler, see Rogačenko, Dokl. Akad. Nauk SSSR (N.S.) **88** (1953), 615–618; MR **14**, 1007; and Mokriščev ibid. **91** (1953), 453–456; MR **15**, 143.]

Finally, he gives a very neat solution for the problem of drawing the common perpendicular to two given skew lines in hyperbolic 3-space. *H. S. M. Coxeter.*

Szász, Paul. Hyperbolische Trigonometrie an dem Poincaréschen Kreismodell abgelesen. Acta Math. Acad. Sci. Hungar. **7** (1956), 65–69. (Russian summary)

In an earlier paper [Acta. Sci. Math. Szeged. **16** (1955), 1–8; MR **16**, 1045] the author gave three lemmas on circles in the Euclidean plane. These are now used to drive the formulae of hyperbolic trigonometry by means of Poincaré's mapping of the hyperbolic plane onto the interior of a circle. *E. Lukacs* (Washington, D.C.).

★ **Hohenberg, Fritz. Konstruktive Geometrie für Techniker.** Springer-Verlag, Wien, 1956. ix+272 pp.

Dies ist ein modernes Lehrbuch der darstellenden Geometrie. Es ist für Studenten der Technik geschrieben; „das technisch wichtige hat den Vorrang vor dem geometrisch wervollen" und es hat die Absicht „mit einem geringen geometrischen Rüstzeug ein Optimum an technisch brauchbarem Wissen und Können zu ermitteln". Das Buch ist schön ausgeführt und die Figuren, vor allem Abbildungen und Skizze technischer Gegenstände, sind ausgezeichnet. Der Text ist gut komponiert, klar, mit mehreren interessanten psychologischen und historischen Bemerkungen. Es gibt drei Abschnitte. A) gibt die Methoden. Mit Normalrisse anfangend werden Geraden und Ebenen, Kreise, Kugeln und Kegelschnitte behandelt, aber immer so dasz die Anschaulichkeit primär ist; es gibt keine Figuren in allgemeiner Lage, keine Riszachse, keine Spuren, aber die geometrische Form eines Eckhauses mit Mansardendach; die Theorie der Kegelschnitte gibt als Beispiele die Mohrsche Sicherkeitsparabel und Betrachtungen über Sonnenuhren. Es folgt ein Kapitel über normale Axonometrie (das Eckhartsche Einschneideverfahren findet man S. 62) und ein solches über schiefe Axonometrie und selbst ein (vom Verfasser herrührenden) Beweis des Pohlkeschen Satzes. Sehr gut wird die Perspektive behandelt, mit schönen Figuren. Dann folgt Photogrammetrisches: Rekonstruktionen aus einer oder zwei Photos. B) gibt Anwendungen auf technisch wichtige Kurven und Flächen, solche zweiter Ordnung, Drehflächen, Schraubflächen, und schlieszt mit einem Kapitel über kotierte Projektion. C) handelt über die Geometrie der Getriebe und Verzahnungen. Es fängt mit der konstruktiven Kinematik der Ebene an, mit vielen technischen Beispiele; die Krümmung der Bahnkurven wird mittels der Annäherung durch eine Trochoidenbewegung gefunden. Es folgen die Verzahnung von Stirnrädern und ein Kapitel über die Konstruktive Geometrie der Bewegungen im Raum. Das Buch bietet die geometrischen Grundlagen des technischen Zeichnens, aber ist auch für den Mathematiker interessant. *O. Bottema* (Delft).

Szász, Pál. Hyperbolic trigonometry developed by means of the Poincaré half-plane. Magyar Tud. Akad. Mat. Fiz. Tud. Oszt. Közl. **6** (1956), 81–85. (Hungarian)

See also: Stubban, p. 1233.

Convex Domains, Extremal Problems, Integral Geometry, Metric Geometry

★**Davenport, H. Le recouvrement de l'espace par des sphères.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 139–145. Georges Thone, Liège; Masson and Cie, Paris, 1956.

Any lattice Λ in Euclidean n-space determines a partition of the whole space into congruent centrally-symmetrical polytopes whose centres are the lattice points. The interior of such a polytope Π, with centre O, consists of all points which are nearer to O than to any other lattice point. Since this "Dirichlet region" or "Voronoi polyhedron" is a fundamental region for the translation group of Λ, it has the same content $d(\Lambda)$ as the unit cell or basic parallelotope. Let s and S denote the in- and circum-spheres of Π, so that s touches the faces nearest to O while S passes through the vertices farthest from O; let $V(s)$, $V(S)$ denote their contents. Such spheres round all the lattice points form a "packing" of density $\delta(\Lambda)=V(s)/d(\Lambda)$ and a "covering" of density $\theta(\Lambda)=V(S)/d(\Lambda)$. Let δ_n denote the upper bound of $\delta(\Lambda)$ for all possible lattices, and θ_n the lower bound of $\theta(\Lambda)$. After remarking that $\delta_{n-2}<2^{-\frac{1}{2}n}n$, the author gives an illuminating sketch of the manner in which the inequality

$$\theta_n>\tfrac{1}{2}-o(1) \qquad (n\to\infty)$$

was obtained by Bambah and Davenport [J. London Math. Soc. **27** (1952), 224–229; MR **14**, 787] and

$$\theta_n<(1.107)^n$$

by G. L. Watson [see the following review].

H. S. M. Coxeter (Toronto, Ont.).

Watson, G. L. The covering of space by spheres. Rend. Circ. Mat. Palermo (2) **5** (1956), 93–100.

In the notation of the preceding review, the author proves that $\theta_n<(1.107)^n$ by applying the ingenious method of Davenport [Rend. Circ. Mat. Palermo (2) **1** (1952), 92–107; MR **14**, 75] to an eight-dimensional lattice consisting of the points whose coordinates $y_1, \cdots, y_8$ are integers satisfying the congruences

$$y_i+y_{i+4}\equiv y_1+y_2+y_3+y_4 \pmod 2 \qquad (i=1, 2, 3, 4).$$

This lattice was discovered by T. Gosset [Messenger of Math. **29** (1899), 43–48, p. 48] as the final member 5_{21} of a finite family of semi-regular $(n+4)$-dimensional polytopes n_{21}, beginning with the triangular prism $(-1)_{21}$. It was rediscovered by E. Cartan [Ann. Mat. Pura Appl. (4) **4** (1927), 209–256, p. 222] in connection with the simple Lie group E_8; he took the coordinates to be nine integers, mutually congruent (mod 3), with sum zero. The same points were shown by H. F. Blichfeldt [Math. Z. **39** (1934), 1–15] to be the centres of the densest regular packing of spheres in 8-space. It also represents the integral Cayley numbers [Coxeter, Duke Math. J. **13** (1946), 561–578, p. 571; MR **8**, 370]. The author's coordinates $y_1, \cdots, y_8$ are essentially the same as those proposed by P. Du Val [Coxeter, Proc. London Math. Soc. (2) **34** (1932), 126–189, p. 185].

The essential step in the present work consists in integrating a certain function over the content of the Dirichlet region which, being the reciprocal of 4_{21} [Coxeter, Regular polytopes, Methuen, London, 1948, p. 204; MR **10**, 261], is dissected by its hyperplanes of symmetry into 192.10! congruent simplexes T_9 [ibid., p. 194]. The author further subdivides T_9 into three orthoschemes, one occupying one-seventh of its content while the other two are congruent. *H. S. M. Coxeter* (Toronto, Ont.).

Molnár, József. On inscribed and circumscribed polygons of convex regions. Mat. Lapok **6** (1955), 210–218. (Hungarian. Russian and English summaries)

The first part of the paper contains the proof of the following two theorems: Theorem 1. If P_n is an n-gon of maximum perimeter inscribed in a convex region, the perimeters satisfy the inequality $P_{n-1}+P_{n+1}\leqq 2P_n$. Theorem 2. If p_n is an n-gon of minimum perimeter circumscribed about a convex region, the perimeters satisfy the inequality $p_{n-1}+p_{n+1}\geqq 2p_n$. In the second part, the author deals with the extension to the hyperbolic plane and to the sphere of these theorems and of Dowker's two analogous results referring to areas [Bull. Amer. Math. Soc. **50** (1944), 120–122; MR **5**, 153]. The problem concerning the perimeter of circumscribed spherical polygons is unsolved. *H. W. Kuhn.*

Florian, A. Eine Ungleichung über konvexe Polyeder. Monatsh. Math. **60** (1956), 130–156.

The author fills the gaps in the proof of an inequality for convex polyhedra due to Fejes Tóth [Canad. J. Math. **2** (1950), 22–31; MR **11**, 386], showing also that equality characterizes the regular polyhedra. *E. G. Straus.*

Busemann, Herbert. Quasihyperbolic geometry. Rend. Circ. Mat. Palermo (2) **4** (1955), 256–269.

Previously [The geometry of geodesics, Academic Press, New York, 1955; MR **17**, 779] the author introduced and gave a characterization of quasihyperbolic geometry, the only two-dimensional Finsler geometry with a transitive group of motions which has not been investigated. A straight two-dimensional G-space (loc. cit. for terminology) is quasihyperbolic if it possesses all translations along two geodesics G, H, where G is a non-parallel asymptote to H. The upper halfplane with the group of conformal mappings which leave the point at infinity fixed may be taken as a model, and in terms of this very complete information is given: The line element has the form $ds=F(dx, dy)y^{-1}$. The geodesics are those solutions of the isoperimetric problem for the Minkowski metric $F(x_1-x_2, y_1-y_2)$ in the (x, y)-plane whose centers lie on the x-axis. The admissible functions F are characterized. "Parallels" behave as in the special case of hyperbolic geometry, but Desargues Theorem holds only in that case. The circles need not be convex, so such spaces do not have convex capsules, let alone negative curvature. Nevertheless, under rather weak restrictions every quasihyperbolic geometry possesses a unique well-behaved angular measure in terms of which the excess of a geodesic triangle is negative and proportional to the area. But these very examples help show that there is no universal (dependent only on the local Minkowski geometry) angular measure for two-dimensional Finsler spaces such that the Gauss-Bonnet theorem always holds.

L. W. Green (Minneapolis, Minn.).

Few, L. The shortest path and the shortest road through n points. Mathematika **2** (1955), 141–144.

The main results of this paper are the following: Theorem 1. Given $n(\geqq 2)$ points in a unit square, there is a path through the n points of length not exceeding $(2n)^{\frac{1}{2}}+1.75$. Theorem 3. Given n points in a unit k-dimensional cube, where k is fixed, there exists a path

through the n points of length not exceeding

$$k\{2(k-1)\}^{(1-k)/(2k)}n^{1-1/k}+O(n^{1-2/k})$$

as $n\to\infty$. The bounds are established by explicit construction of alternative paths through the points (2 paths in Theorem 1, 2^{k-1} paths in Theorem 3), then bounding the sum of the lengths of these paths. These results improve bounds obtained by Verblunsky [Proc. Amer. Math. Soc. 2 (1951), 904–913; MR 13, 577]. The author also generalizes a construction of L. Fejes Tóth [Math. Z. 46 (1940), 83–85; MR 1, 263] to show: Theorem 5. If k is fixed and $n\to\infty$ there exist n points in a unit k-dimensional cube for which the length of the shortest path exceeds, or is asymptotically equal to

$$\{\Gamma(1+\tfrac{1}{2}k)\}^{1/k}\pi^{-\frac{1}{2}}n^{1-1/k}.$$

H. W. Kuhn (Bryn Mawr, Pa.).

Legrady, Kurt. Symplektische Integralgeometrie. Ann. Mat. Pura Appl. (4) **41** (1956), 139–159.

This is a study of the integral geometry in a symplectic manifold in the sense of Siegel [Amer. J. Math. **65** (1943), 1–86; MR **4**, 242]. The main concern is a Crofton-type formula on the measure of geodesics which meet a compact subset of a hypersurface. The symplectic group which acts on the manifold does not permute the geodesics transitively. However, it is proved that a density exists on each domain of transitivity of the geodesics. A density also exists for the set of geodesics which belong to a domain of transitivity and which pass through a given point. A combination of these two results leads to a Crofton-type formula. In the development the author used and extended some work of the reviewer on integral geometry of homogeneous spaces [Ann. of Math. (2) **43** (1942), 178–189; MR **3**, 253]. *S. Chern* (Chicago, Ill.).

Differential Geometry

Marussi, Antonio. Sui parametri differenziali del campo gravimetrico. Rend. Sem. Mat. Fis. Milano **25** (1953–54), 15–17 (1955).

In geodetics and in geophysics the scalar field of the module of the vector of gravitation is important because of its relatively easy experimental determination. So it is interesting to investigate the relations between the vector field of the gravitation and the scalar field of the module of this vector. The author recalls some well-known results in this domain and indicates a method, suitable for further applications. *H. Bremekamp* (Delft).

★ Marchaud, André. Propriétés différentielles des courbes et des surfaces d'ordre borné. Colloque sur les questions de réalité en géométrie, Liège, 1955, pp. 39–57. Georges Thone, Liège; Masson & Cie, Paris, 1956. 250 fr. belges; 1900 fr. français.

This is mainly a survey of the author's papers on continuums and surfaces of bounded real order, in particular on surfaces of order three. The reader is referred to the reviews of the four last papers [MR **1**, 169; **8**, 571; **14**, 1118]. — A mistake in the first of these four papers is corrected by showing that a ruled surface S of order three has a continuous tangent plane everywhere with the possible exception of the singular straight line of S. The paper ends with some — apparently new — results on the existence and continuity of the Euler-Dupin indicatrix on S. *P. Scherk* (Philadelphia, Pa.).

Bouligand, Georges. Types d'intégrales généralisées pour une classe d'équations aux derivées partielles du premier ordre. C. R. Acad. Sci. Paris **242** (1956), 2423–2426.

Dans l'espace euclidien cartésien R^3, une intégrale contingente de l'équation $p^2+q^2=1$ est définie comme une fonction lipschitzienne $z=\varphi(x, y)$ (dont le domaine de définition n'est pas précisée) telle qu'en chaque point M de la surface S: $z=\varphi(x, y)$ aucune demi-tangente ne soit située à l'intérieur du cône de l'équation en M, mais qu'au moins une soit située sur lui. La distance d'un point à un ensemble fermé est, en dehors de l'ensemble, une intégrale contingente. Toute surface intégrale contingente possède la propriété suivante: Le rapport de l'aire sur S d'un ensemble mesurable de S à l'aire de sa projection horizontale est $=\sqrt{2}$. Les surfaces intégrales aréolaires sont définie par cette propriété. Une surface moulure à plan directeur horizontal dont le profil P est de la forme $v=\int_0^u \varepsilon(t)dt$, $\varepsilon(t)$ étant une fonction mesurable borné prenant presque partout les valeurs $+1$ et -1 est une surface intégrale aréolaire. Si en un point de P les pentes de toutes les demi-tangentes à P sont en valeur absolue <1, la surface moulure est une intégrale aréolaire sans être une intégrale contingente. Ces considérations sont généralisées par remplacement de la métrique euclidienne par une métrique finslérienne, les aires étant entendues au sens de G. Choquet.

C. Y. Pauc (Lafayette, Ind.).

Giger, Hans. Beiträge zur Theorie von Stützfunktion und Radius — Die Radialflächen. Comment. Math. Helv. **30** (1956), 241–256.

Let x be the position vector of a surface and N the corresponding normal; and $r=\sqrt{x^2}$ the 'radius' and $p=-(N, x)$ the 'supportfunction'. The author defines as 'radial' those surfaces for which there exists a functional relationship between r and p. The surfaces of revolution are, of course, a particular case of the radial surfaces.

In the present paper the author proves the following, 1) A regular and analytic convex closed surface which is at the same time radial is necessarily a surface of revolution. 2) A radial surface of constant mean curvature is necessarily a surface of revolution. It follows from this that the only minimal surface which is at the same time radial is the catenoid.

The author shows finally that there occur on radial surfaces, which are not surfaces of revolution, umbilical lines which do not belong to the net of curvature lines. Hence, there exist radial surfaces on which the umbilical lines are neither plane nor spherical curves. *R. Blum.*

★ Hristow, W. K. Die Gaussschen und geographischen Koordinaten auf dem Ellipsoid von Krassowsky. Verlag Technik, Berlin, 1955. 254 pp. DM 41.00.

This book is an expanded version of the author's monograph Die Gauss-Krügerschen Koordinaten auf dem Ellipsoid [Teubner, Leipzig-Berlin, 1943] enlarged to include sufficient background material to make it suitable as a text. The earlier work was essentially a collection of the author's papers on Transverse Mercator (called "Gauss-Krüger" in Germany, and simply "Gauss" in the present work) map coordinates appearing in Z. Vermessungswesen from 1933 to 1943. In these, Hristow made a very important contribution to the geodetic literature by developing the various standard formulas (viz., coordinate transformation, meridian convergence, scale factor, distance and angle reduction, and forward and

inverse geodetic problems) in forms suitable for computation by calculating machines, which have subsequently proved to be adaptable for electronic high-speed calculators.

The Transverse Mercator projection has become the most universally adopted system of map coordinates for geodetic work. Hristow's book gives the most complete presentation of the theory of these coordinates available up to now. The emphasis is slanted toward the practical application, as is indicated by the extensive use of Taylor series and numerical quadratures. However, this approach is hampered by a total lack of tables. (Most of the requisite tables can be found in the various manuals of the U. S. Army Map Service on the Universal Transverse Mercator Projection.)

Chapter I discusses the elementary differential geometry of an arbitrary surface using the classical Gauss notation instead of the more modern tensor approach. In Chapter II, this material is applied to an ellipsoid of revolution. The forward and inverse problems are solved here in terms of geographic coordinates. Chapter III introduces Soldner, Mercator, and Gauss (Transverse Mercator) coordinates with reference to the sphere. The various standard formulas mentioned above are derived in spherical terms. Chapter IV briefly discusses the Soldner coordinates with respect to the ellipsoid.

The heart of the book is Chapter V in which ellipsoidal Gauss coordinates (x, y) are discussed. These are first completely defined by the following two properties: 1. They are isometric; 2. On the central meridian, $y=0$, and x=the length of the meridian (counted from some starting point, $x=0$). This is based on the well-known theorem of function theory that an analytic function is completely defined in a region of no singularities if its value is specified on any arc within the region. Then the relations between Gauss and Mercator, and Gauss and geographic coordinates are obtained to the 8th order. In addition to the other standard formulas (meridian convergence, etc.) there are also derived the relation between Soldner and Gauss coordinates, the Gauss projection of the geodetic line, and the transformation between sets of Gauss coordinates referred to different central meridians and an arbitrary origin.

Chapter VI is appendical in nature. In the first part, the Krassovsky ellipsoid parameters are stated; in the second part, an interpolation scheme valid to third differences is developed.

Despite the book's thoroughness and detail, it cannot be considered as comprehensive in its field, since it completely omits mention of the work of K. Ludwig [J. Reine Angew. Math. **185** (1943), 193–230; MR **7**, 28], who approached the same subject from an entirely different viewpoint by making use of elliptic functions and their generalizations, instead of infinite series.

It should be pointed out that the title of this work is misleading. The only use made of the Krassovsky ellipsoid is simply to push it forward as a world reference ellipsoid in place of the Hayford ellipsoid, generally accepted outside the Soviet orbit. Reference to the Krassovsky ellipsoid appears in only two pages of the text.

There is no index. Otherwise the format and typography are acceptable. However, they fall far below the standards set by a comparable work by König and Weise [Die Mathematische Grundlagen der höheren Geodäsie und Kartographie, Bd. I, Springer, Berlin, 1951; MR **13**, 157].

Misprints noted: p. 78; (13): $d\eta^2/dt$ should be $d\eta^2/d\phi$; p. 204, (8): should be $d_{12}^2=\Delta x^2+\Delta y^2$. In addition it should be noted that 1.0, 2.0, etc. stand for 1st order, 2nd order, etc. *B. Chovitz* (Washington, D.C.).

Saban, Giacomo. Qualche osservazione sulle rigate a piano direttore chiuse. Arch. Math. **7** (1956), 214–218.

A ruled surface given by the equations: $\mathbf{x}=\mathbf{d}(u_1)+u_2\mathbf{g}(u_1)$ is said to have a closed plane directrix if $\mathbf{g}\cdot\mathbf{k}=0$ where $\mathbf{k}$ is a constant vector and if the spherical indicatrix of $\mathbf{g}$ is closed. For such a surface the author shows that there are at least two generators along which the tangent plane remains fixed. The surface is "proper', if the indicatrix of $\mathbf{g}$ is a circle, covered exactly once. It is shown that a proper surface has at least four generators along which the osculating congruence is stationary. These results are generalizations of the four-vertex theorem for plane curves. *C. B. Allendoerfer* (Seattle, Wash.).

Mutô, Yosio. On convex mapping. Tôhoku Math. J. (2) **7** (1955), 141–145.

A mapping f of class C^2 of an n-dimensional compact manifold M^n ($n\geqq 2$) of class C^2 into the $(n+1)$-dimensional euclidean space E^{n+1} is called convex if $\sum \Omega_{jk}v^jv^k\leqq 0$ for any v^j ($j, k=1, 2, \cdots, n$) and $\sum \Omega_{jk}v^jv^k<0$ for some v^j, where Ω_{jk} is the second fundamental tensor of $f(M^n)$ for a suitably chosen direction of the normal. On these assumptions the author proves that f is an homeomorphism and that M^n is homeomorphic to an n-sphere. *L. A. Santaló* (Buenos Aires).

Ohshio, Shigeru. Volume, surface-area and total-mean-curvature. Sci. Rep. Kanazawa Univ. **4** (1955), no. 1, 21–28.

G. Bol made interesting uses of the internal parallel surfaces of a closed convex surface [Abh. Math. Sem. Sem. Hansischen Univ. **15** (1943), 27–36, 37–56; MR **7**, 475, 474]. Related with this work the author gives and discusses some differential and integral formulas referring the enclosed volume, the surface area and the mean curvature of such parallel surfaces. *L. A. Santaló.*

Wintner, Aurel. On Weyl's identity in the differential geometry of surfaces. Ann. Mat. Pura Appl. (4) **41** (1956), 257–268.

This paper contains remarks and complements on Weyl's identity in the theory of surfaces. Let H and K be the mean curvature and the Gaussian curvature of a surface and let $J=(H^2-K)^{\frac{1}{2}}$. Let ∇', Δ' denote the first and second Beltrami differential operators with respect to the first fundamental form, and ∇'', Δ'' those with respect to the second fundamental form. Then Weyl's identity is

$$\Delta''H-\tfrac{1}{2}\Delta'K-2KJ^2=\{\}/J,\quad K\neq 0,$$

where $\{\}=2\nabla''(H, J)-\nabla'(K, J)$. The inequality which follows from it plays a vital rôle in the global imbedding problem of closed convex surfaces. Both the identity and the resulting inequality were proved only at non-umbilical points. The author transforms the identity to a form to be valid also at an umbilic and supplies an argument for the validity of the inequality at an umbilic. When the first fundamental form is replaced by the third fundamental form, the author derives an identity which has as a consequence an inequality of Miranda. The last sections contain some discussions of differentiability properties at umbilics and applications of the identities. *S. Chern.*

Suguri, Tsuneo; and Ueno, Seitaro. Normal coordinates in Kählerian spaces. Mem. Fac. Sci. Kyusyu Univ. Ser. A. **9** (1956), 83–99.

Using normal coordinates about a point 0 in a Kählerian space, it is proved that the fundamental differential forms are completely determined by the numerical values of the curvature tensor and its successive covariant derivatives at 0. A Kählerian space is called symmetric, if a symmetry about any point is an isometry. It is proved that a necessary and sufficient condition that a Kählerian space be symmetric is that the covariant derivatives of the curvature tensor be zero. *S. Chern* (Chicago, Ill.).

Haefliger, André. Sur les feuilletages analytiques. C. R. Acad. Sci. Paris **242** (1956), 2908–2910.

Let V be a real analytic manifold of dimension n with an analytic leaved structure of codimension 1. A closed transversal is by definition a differentiable map of a circle into V such that at every image point its tangent vector is transversal to the leaf through it. The author states the fundamental lemma: If the leaved structure has a closed transversal, the element of the fundamental group generated by the latter is of infinite order. From this the following theorems are deduced: I. If the fundamental group of V has only elements of finite order, then V is not compact and every leaf is closed. II. If the fundamental group of V has an infinite cyclic group of finite index, then there is at least one compact leaf and there is no leaf which is locally everywhere dense. Finally, a theorem is stated on the existence of integrating factor of a completely integrable analytic Pfaffian form. Proofs are sketched. *S. Chern* (Chicago, Ill.).

Čech, Eduard. Deformazioni proiettive nel senso di Fubini e generalizzazioni. Confer. Sem. Mat. Univ. Bari no. **9** (1955), 12 pp.

An expository lecture, without derivations, on projective deformations of surfaces in three-dimensional projective spaces in the sense of Fubini and on a generalization of that to ∞^1 systems of surfaces.

C. C. Hsiung (Bethlehem, Pa.).

Barner, Martin. Zur Kinematik der Projektivabwicklung von Raumkurven. Math. Nachr. **14** (1955), 1–16.

In a projective space a curve is called a complex curve if its tangents belong to a linear complex. Let a, b, c be the invariants of a curve K with respect to an arbitrary parameter in a three-dimensional projective space R_3. Associated with every point of K there is a complex curve such that all the ∞^1 complex curves are transformed into each other by a projective motion. The curve K is an envelope of the family of the complex curves, and has a fourth order contact at each point with a complex curve. The complex curves form a family of asymptotic curves of the surface F generated by the complex curves. The other family of asymptotic curves of F maps the complex curves projectively onto each other, and defines a point correspondence between the complex curves and the curve K such that at each pair of corresponding points the curve K and the complex curve have the same invariants a, c and for the complex curves $b=0$. Similar results in Lie geometry are deduced for a surface and a family of spherical curves in ordinary Euclidean space.

C. C. Hsiung (Bethlehem, Pa.).

Barner, Martin. Geradengeometrische Kennzeichnung der W-Kurven des projektiven Raumes. Arch. Math. **6** (1955), 462–470.

It is proved that in a projective space R_n of an odd dimension $n \geqq 5$ a curve C with all of its osculating linear spaces of dimension $\frac{1}{2}(n-3)$ lying on a fixed hyperquadric Q but not in any hyperplane admits a one-parameter group of projective transformations leaving C and Q invariant, if and only if in a fixed null system every point of C corresponds to the osculating hyperplane of C at the point. Furthermore, in R_n a W-curve C (a curve is called a W-curve if it admits a one-parameter group of projective transformations preserving itself), together with all of its osculating linear spaces of dimension $\frac{1}{2}(n-3)$, lies on a fixed hyperquadric if in a fixed null system every point of C corresponds to the osculating hyperplane of C at the point. *C. C. Hsiung.*

Segre, Beniamino. Intorno ad un problema di Wilhelm Blaschke. Abh. Math. Sem. Univ. Hamburg **20** (1955), 28–40.

Let F be a surface in a four-dimensional projective space S_4. A hyperplane S_3 in S_4 is said to be of Blaschke for F, if it intersects the tangent planes of F in the lines of a W congruence with nondegenerate and distinct focal surfaces. In connection with a problem proposed by Blaschke [Rend. Circ. Mat. Palermo (2) **3** (1954), 193–197; MR **16**, 854] the following results are obtained. A surface F can have only 0, 1, 2, 3, 4 or 5 linearly independent hyperplanes S_3 of Blaschke. No surface F can have three hyperplanes of Blaschke belonging to a same pencil. If a surface F has four or five linearly independent hyperplanes of Blaschke, then in any linear system determined by these hyperplanes no any other hyperplanes can be of Blaschke for F. Every surface F having four linearly dependent hyperplanes of Blaschke admits altogether ∞^1 such hyperplanes which are tangent to a twice degenerate quadric cone in S_4. Finally, parametric equations in terms of two arbitrary functions of one variable alone are obtained for the most general surface F having ∞^1 hyperplanes of Blaschke. *C. C. Hsiung.*

Laugwitz, Detlef. Grundlagen für die Geometrie der unendlichdimensionalen Finslerräume. Ann. Mat. Pura Appl. (4) **41** (1956), 21–41.

Using techniques developed in his earlier papers on differential geometric methods not limited to finitely-dimensional spaces [Math. Z. **61** (1954), 100–118, 134–149; MR **16**, 512] the author develops the foundations of a theory of infinitely dimensional Minkowski and Finsler spaces. A preliminary paragraph is devoted to a suitable tensor calculus, and a form of Taylor's theorem is derived for the Fréchet differentials. A Minkowski space is defined to be a real Banach space $B=\{\xi^t\}$ in which a positive definite functional $F(\xi)$ is defined, where $F(\xi)$ is positively homogeneous of the first degree and satisfies the triangle inequality. It is furthermore assumed that F defines a norm of the norm topology of B such that for certain positive numbers M, M' the inequality $(M')^{-1}\cdot\|\xi\| \leqq F(\xi) \leqq M\cdot\|\xi\|$ holds. This ensures that the function of support $H(u)$ (defined in a manner analogous to the finitely dimensional case) and $F(\xi)$ are continuous. Certain differentiability conditions are imposed on F and H, and the quadratic form whose coefficients are second derivatives of F is supposed to be positive definite. An infinitely dimensional differentiable manifold is called a Finsler space if at each point (x^t) a Minkowski metric $F(x, \xi)$ is defined in the tangent space. Subspaces of a Minkowski space are then Finsler spaces. The equations of the geodesics of a Finsler space are derived and the notion of the osculating Riemannian space is introduced.

Following a method used in the finitely dimensional case [Arch. Math. 6 (1955), 448–453; MR 17, 408] the author obtains generalisations of the covariant derivatives defined (for the finitely-dimensional case) by E. Cartan, W. Barthel and the reviewer. *H. Rund* (Durban).

See also: Bouligand, p. 1196; Busemann, p. 1235; Longo, p. 1240; Bauer, p. 1256.

Riemannian Geometry, Connections

★**Spencer, D. C. On Green's operators.** The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1952. i+23 pp. (mimeographed)

Ce cours est consacré à l'opérateur de Green sur une variété riemannienne V „uniformly smooth" et à quelques applications (dans le cas où V est compacte). Une variété riemannienne V est dite „uniformly smooth" s'il existe une constante positive η telle que tout point x de V soit centre d'une sphère géodésique $S_\eta(x)$ de rayon η et si le tenseur métrique et les dérivées partielles, jusqu'au 4ième ordre, de ses composantes en coordonnées normales de centre x, sont bornées dans $S_\eta(x)$ indépendamment de x. Une variété uniformly smooth est complète et de bord négligeable. Soit N^p l'espace de Hilbert des p-formes de V de normes finies, N_r^p le sous-espace des p-formes de classe C^r. Si $B^p=\overline{dN_1}^{p-1}$, $*B^p=\overline{\delta N_1}^{p+1}$, N^p admet la décomposition en somme directe de sous-espaces orthogonaux $N^p=B^p+*B^p+H^p$, où H^p est le sous-espace des p-formes harmoniques de normes finies. En particulier tout élément de H^p est fermé et cofermé; $H\alpha$ désignera la composante harmonique de $\alpha \in N^p$. L'A. généralise alors aux variétés uniformly smooth [voir aussi Spencer, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 533–534; MR **15**, 647] les résultats de Milgram et Rosenbloom [ibid. **37** (1951), 180–184; MR **13**, 160] concernant les solutions de l'équation de la chaleur $\Delta\varphi+\partial\varphi/\partial t=0$, où φ est une p-forme dépendant du paramètre t. Il existe pour $t>0$ un opérateur W_t tel que, pour $\alpha \in N^p$, $W_t\alpha$ satisfont à l'équation de la chaleur, avec $W_0\alpha=\alpha$. L'opérateur W_t est symétrique, définit un semi-groupe, commute avec d et δ, et tend vers l'opérateur H quand $t\to\infty$. Si Z^p est un cycle à support compact, la période correspondante de $W_t\alpha$ ($t>0$) est indépendante de t. Il en résulte que $H\alpha$ est l'unique forme harmonique ayant les mêmes périodes que α pour tout cycle à support compact.

Un opérateur de Green sur V est un opérateur G satisfaisant à $\Delta G\varphi=\varphi-H\varphi$, $HG\varphi=0$ pour toute p-forme φ de classe C^∞ à norme finie. Si $G_t=\int_0^t(W_t-H)\delta\tau$ ($HG_t=0$), pour qu'il existe sur V un opérateur de Green, il faut et il suffit que G_t converge quand $t\to\infty$. On sait que, si V est compacte, G existe. L'A. indique ici deux démonstrations de ce résultat, dont la seconde repose sur la considération de l'équation $\alpha-W_t\alpha=\beta$, qui est une équation de Fredholm à noyau symétrique, régulier.

Par l'opérateur de Green, l'A. esquisse aussi la méthode d'approche consistant à étudier, par des procédés classiques, l'équation $\Delta_s\alpha=\beta$ (avec $\Delta_s=\Delta+s$; $s>0$). Il existe pour tout $s>0$, $\beta \in N_\infty^p$, un opérateur G_s satisfaisante à $\Delta_sG_s\beta=\beta-H\beta$. Si la variété est uniformly smooth, G_s est symétrique, commute avec Δ_s et $HG_s=0$. Il convient alors d'étudier dans quelles circonstances G_s converge pour $s\to0$.

Le papier se termine (pour illustrer les usages de l'opérateur de Green) par un exposé faisant appel à G des deux théorèmes de H. Cartan liés aux problèmes de Cousin [Séminaire de topologie algébrique de l'Ecole Normale Supérieure, 1950–51; MR **14**, 670] V étant alors une variété kählerienne compacte. *A. Lichnerowicz.*

Vrănceanu, G. Sur les espaces à courbure constante. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 59–64. (Romanian. Russian and French summaries) Let

$$(1)\qquad ds^2=\alpha^2\sum(dx^i)^2+\beta\sum x^idx^i,$$

where α and β are functions of $r=\sqrt{\sum}(x^i)^2$, be the metric of a Riemannian space V_n. It is shown that, if (1) has constant curvature k, then β is given by the formula:

$$(2)\qquad \beta=(\alpha'^2+2\alpha\alpha'r^{-1}+k\alpha^4)/(1-k\alpha^2r^2),$$

where α' is the derivative of α with respect to r. The author shows then how one can obtain from (1) different classical canonical forms of Riemannian spaces with constant curvature. The proof of formula (2) is based on the introduction of a system of n orthogonal congruences in V_n and the calculation of the components of the curvature tensor with respect to that system.

R. Blum (Saskatoon, Sask.).

See also: Conner, p. 1197; Kondo, p. 1246.

Complex Manifolds

Guggenheimer, Heinrich. Modifications of Kähler manifolds. Ann. Mat. Pura Appl. (4) **41** (1956), 87–93.

A complex manifold V' is called a modification of a complex manifold V at a submanifold $P\subset V$ if there is an analytic submanifold $P'\subset V'$ and an analytic homeomorphism of $V'-P'$ onto $V-P$. As an immediate consequence of the cohomology sequences of the pairs (V, P) and (V', P'), it follows that $\chi(V)-\chi(P)=\chi(V')-\chi(P')$; in the particular case of a dilation, this yields a formula of Segre [same Ann. (4) **37** (1954), 139–155; MR **16**, 511]. By applying the various representations which he has obtained for the de Rham cohomology groups of a Kähler manifold, the author shows that the geometrical r-genera (the number of linearly independent holomorphic r-differentials) are invariant under any modifications in which V and P are both Kähler manifolds; the usefulness of these methods is further illustrated by the determination of the Betti numbers of V' from those of V, P, P', for V and V' of complex dimension 3. *R. C. Gunning.*

Hawley, N. S. Complex bundles with Abelian group. Pacific J. Math. **6** (1956), 65–82.

This paper is concerned with certain complex bundles whose bundle groups are abelian and with some bundle spaces of such bundles. A general theory is developed by analogy with the Kodaira-Spencer theory. It specializes to the latter theory and also to the case of G-bundles, where G is the additive group of complex numbers mod (ω_1, ω_2), ω_1 and ω_2 being complex numbers with ω_1/ω_2 not real. Among miscellaneous results, it is shown, by a construction involving Hopf manifolds, that "compact Kähler manifolds form a very small subclass of the compact complex manifolds." *S. S. Cairns* (Urbana, Ill.).

Algebraic Geometry

Longo, Carmelo. Sul modello minimo della varietà degli elementi del 2° ordine di S_r. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 614–618.

This is a preliminary note of the principal results contained in the paper reviewed below. *J. G. Semple.*

Longo, Carmelo. Gli elementi differenziali del 2° ordine di S_r. Rend. Mat. e Appl. (5) **13** (1955), 335–372.

In this paper the author studies in detail the aggregate of E_2 of S_r and largely rounds off investigations (particularly for the case $r=2$) which were first initiated by E. Study [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Cl. **53** (1901), 338–403]. A non-special E_2 of S_r is characterized by its fundamental form (centre P, tangent line p, osculating plane π) and by its "curvature" (suitably interpreted): but the aggregate of such E_2 has to be closed by adding to it (a) the inflexional E_2, each of which has zero curvature and no definite osculating plane, and (b) the cuspidal E_2, each of which has a well-defined osculating plane but infinite curvature. It is shown then that every element of the extended aggregate can be given unambiguous coordinates analogous to those defined by Study for E_2 in the plane; and this procedure leads forthwith to the construction of a minimum model W of all E_2 of S_r. There are sub-varieties I and C of W, of dimensions $2r-1$ and $3r-3$, which map respectively the inflexional and the cuspidal E_2 of S_r; and it appears then that W is simply generated by ∞^{2r-1} spaces Σ_{r-1} each of which joins a point of I to a homologous generating space Σ_{r-2} of C. Among other results, the author finds for example that in the case $r=3$ the model W is a 7-fold variety of order 87,570 in [659]. For the subordinate model $\overline{W}$ of all the E_2 of S_r with centre at a fixed point O an analogous geometric characterization and an explicit formula for the order are obtained.

The first section of the paper is devoted of necessity to an extensive preliminary study of the minimum model $M(0, 1, 2; r)$ of all incidence forms (P, p, π) of S_r. There is also, towards the end, some interesting discussion of an alternative approach to the construction of W by use, as an intermediate stage, of the Grassmannian of osculating planes of all the E_2 on the Veronesean $V(2, r)$ of quadrics in S_r. The idea behind this alternative approach would appear to be applicable also to the study of the E_k of S_r for $k>2$. *J. G. Semple* (London).

Godeaux, Lucien. Il teorema di Picard sulla regolarità del sistema aggiunto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 265–266.

Creangă, I. Sur une correspondance entre les congruences de droites. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 135–142. (Romanian. Russian and French summaries)

Let K_1 and K_2 be two rectilinear congruences and Σ a surface in the Euclidean 3-space. The generators g_1 of K_1 and g_2 of K_2 are said to be in correspondence if they meet in a point M of Σ. Let γ be a curve on Σ passing through M. Let R_1 in K_1 and R_2 in K_2 be the ruled surfaces which pass through γ. On the corresponding generators g_1 in R_1 and g_2 in R_2 one can then establish a point correspondence in the following manner: M_1 on g_1 and M_2 on g_2 will correspond to each other if the tangent planes of R_1 in M_1 and R_2 in M_2 intersect in a straight line situated in the tangent plane of Σ in M. It is shown that this latter correspondence is an homology. If the direction of the tangent to γ in M is changed, the center of the homology describes a conic determined by the foci of g_1 in K_1 and of g_2 in K_2 and the characteristic point of the plane (g_1, g_2). *R. Blum* (Saskatoon, Sask.).

Godeaux, Lucien. Sur la structure des points de diramation d'une surface multiple, d'ordre 157. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 413–418.

Spampinato, Nicolò. Sulle falde di Halphen. Ricerche Mat. **4** (1955), 191–206.

A detailed study of two dimensional branches introduced by Halphen. By using the tri-dual algebra over the complex numbers, an extension is made to an eight dimensional branch in S_{11}. *G. B. Huff* (Athens, Ga.).

Spampinato, Nicolò. Sull'invarianza birazionale dell'intersezione di una falda bidimensionale con una superficie. Ricerche Mat. **4** (1955), 126–149.

For a specific example, it is verified by computation that certain intersection multiplicities for a two dimensional branch and a surface in complex S_3 remain invariant under a cubic Cremona transformation.

G. B. Huff (Athens, Ga.).

Kreiss, Heinz-Otto. Über syzygetische Flächen. Ann. Mat. Pura Appl. (4) **41** (1956), 105–111.

Con questo lavoro l'Autore porta un notevole contributo al problema del massimo numero di punti doppi isolati di cui possa esser dotata una superficie algebraica di dato ordine appartenente allo spazio ordinario. Egli considera superficie sizygetiche di S_3, e cioè quelle superficie di S^3 che hanno come punti singolari gli n^3 punti base di una rete Σ di superficie d'ordine n; e scegliendo in modo opportuno la rete Σ riesce a costruire esempi di superficie algebriche d'ordine $2n$ aventi almeno $\frac{1}{2}(5n^3-3n^2)$ singolarità isolate per n qualunque, e superficie algebriche d'ordine $2n$ (con $n\leqq 7$) $3n^3-2n^2$ singolarità isolate, migliorando sensibilmente (per $n\geqq 4$) i risultati già noti sull'argomento. *D. Gallarati* (Genova).

Longo, Carmelo. Sul modello minimo degli elementi cuspidali del piano. Ann. Scuola Norm. Sup. Pisa (3) **9** (1955), 45–63.

In this paper the author studies the varieties represented by the cuspidal elements E_6 and E_7, that is, by the curves having a same cusp with points of intersection respectively in groups G_7 and G_8 of coincident points at the cusp. The Grassmann coordinates of the totality of the curves representing a given element are introduced and used to determine the coordinates of a special element so that the equations of the variety represented by this special element can be written out. Finally, the discussion is extended to a larger class of elements $E^{(\nu,\mu)}$ defined by the curves passing through a same point 0 and having a unique branch of order ν and class μ with points of intersection in a group G_k of coincident points at 0, where $k=\nu(\nu+\mu)+\delta-1$, δ being the greatest common divisor of μ and ν. *C. C. Hsiung* (Bethlehem, Pa.).

NUMERICAL ANALYSIS

★**Desoer, C. A. The iterative solution of networks of resistors and ideal diodes.** Proceedings of the National Electronics Conference, Chicago, 1955, Vol. 11, pp. 678–685, National Electronics Conference, Inc., Chicago, Ill., 1956.

In the present paper it is shown that a modification of the conjugate gradient method [see Hestenes and Stiefel, J. Res. Nat. Bur. Standards **49** (1952), 409–436; MR **15**, 651] can be used to solve nonlinear problems arising in the study of networks of resistors and ideal diodes. The method is illustrated by an example and is applicable to other situations. *M. R. Hestenes.*

★**Bibliography.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 283–370. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

This bibliography has four parts. I. Monte Carlo proper (75 entries with 4 addenda). II. Generation and testing of random digits and known sources of random digit tables (42 entries). III. Articles and books, mostly without abstracts, in which sampling is primarily relied upon for a solution or verification (91 entries). IV. Selection of articles and books on or making use of stochastic processes, ··· mostly without abstracts (170 entries). Most of these entries are accompanied by (sometimes fairly lengthy) descriptive remarks.

★**Curtiss, J. H. A theoretical comparison of the efficiencies of two classical methods and a Monte Carlo method for computing one component of the solution of a set of linear algebraic equations.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 191–233. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

This article was reviewed earlier in MR **16**, 1156 as a report [Inst. Math. Sci. New York Univ. Rep. **IMM-NYU** 211, (1954)].

★**Hastings, Cecil, Jr.; Hayward, Jeanne; and Wong, James P., Jr. Approximations in numerical analysis: a report on a study.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 77–81. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

Bracken, Robert H. A general curvefitting subroutine for transcendental functions. Computers and Automation **5** (1956), no. 7, 16–18.

Booth, Andrew D. A computer program for finding roots. Computers and Automation **5** (1956), no. 7, 20–21.

Schubart, J. Numerische Aufsuchung periodischer Lösungen im Dreikörperproblem. Astr. Nachr. **283** (1956), 17–22.

★**Frankel, S. P. Some qualitative comments on stability considerations in partial difference equations.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 73–75. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

The author comments on the numerical solution by finite-difference methods of partial differential problems of the "initial-value" type. The difference approximation may be "unstable" or "stable" [see O'Brien, Hyman, and Kaplan; J. Math. Phys. **23** (1951), 223–251; MR **12**, 751], depending on whether errors are magnified or not. The author briefly discusses the stability of the most straight-forward difference approximations to $y_t=ay_{xx}$, $y_{tt}=c^2y_{xx}$, and $y_t=cy_x$. He shows that sometimes the restrictions on Δx, Δt imposed to ensure stability can be interpreted in physical and/or mathematical terms. *M. A. Hyman* (Philadelphia, Pa.).

Douglas, Jim, Jr.; and Gallie, T. M., Jr. Variable time steps in the solution of the heat flow equation by a difference equation. Proc. Amer. Math. Soc. **6** (1955), 787–793.

The authors consider the numerical solution of

$$(*)\qquad \begin{aligned} &u_t=u_{xx} && (0<x<1;\ t>0),\\ &u(0,t)=u(1,t)\equiv 0 && (t\geqq 0),\\ &u(x,0)=f(x) && (0<x<1), \end{aligned}$$

where

$$f(x)=\sum_1^\infty a_n \sin n\pi x,\quad \sum_1^\infty |a_n|n^4<\infty.$$

In the numerical solution, (*) is replaced by

$$\frac{w(x,t)-w(x,t-\Delta t)}{\Delta t}=\frac{w(x+\Delta x,t)-2w(x,t)+w(x-\Delta x,t)}{\Delta x^2},$$

where Δx is constant but $\Delta t>0$ may vary with t. This "backward" difference equation is known to be stable (i.e., errors do not grow) for all Δx, Δt [see O'Brien, Hyman, and Kaplan, J. Math. Phys. **23** (1951), 223–251; MR **12**, 751]. For both Δx, Δt constant, the maximum error $E\equiv u-w$ in the finite region R: $0\leq x\leq 1$, $0\leq t\leq T$ is $O(\Delta x^2)$ as $\Delta x\to 0+$. The authors extend this result to two cases of variable Δt:

$$\Delta t_k=(\alpha+\beta t_k)\Delta x^2,\quad \Delta t_k=e^{\gamma t_k}\Delta x^2;\quad (\alpha>0,\ \beta>0,\ \tfrac{1}{2}\pi^2>\gamma>0).$$

Here Δt_k is Δt after k steps in the positive t-direction. The reviewer has shown (unpublished) that $E=O(\Delta x^2)$ assuming only that $r=\Delta t/\Delta x^2$ is bounded in R as $\Delta x\to 0$; the variation of r as a function of t $(0\leq t\leq T)$ can be arbitrary. For convergence of the difference solution ($E\to 0$ as $\Delta x\to 0$), the reviewer finds that $\max_R r=o(1/\Delta x^2)$ or $\max_R \Delta t=o(1)$ is sufficient (the best possible result). *M. A. Hyman* (Philadelphia, Pa.).

Douglas, Jim, Jr.; and Gallie, T. M., Jr. On the numerical integration of a parabolic differential equation subject to a moving boundary condition. Duke Math. J. **22** (1955), 557–571.

The authors discuss numerical solution of the following Stefan's problem:

$$u_t=u_{xx}\quad (0<x<\xi(t)),\quad u_x(0,t)=-1\quad (t>0),$$

$$u(\xi,t)=0\ (t\geqq 0),\ \frac{d\xi}{dt}=-u_x(\xi,t)\ (t>0),\ \xi(0)=0.$$

The authors propose two finite-difference procedures for solving this non-linear problem, each involving an implicit difference approximant to $u_t=u_{xx}$. One, involving variable Δt, is discussed in detail and is shown to

be stable numerically and uniformly convergent (as Δx, $\Delta t \to 0$) to the solution of the differential problem in the region

$$0 \leqq x \leqq \bar{x},\ 0 \leqq t \leqq \bar{t},$$

where $\bar{x} = \xi(\bar{t}) < 1$. Sample calculations using this method appear to give good results. *M. A. Hyman.*

See also: Heinrich, p. 1175; Mel'nik, p. 1190; Müller, p. 1215; Babuška et al, p. 1253.

Tables

Thomas, D. E. Tables of phase of a semi-infinite unit attenuation slope. Bell. Tel. System Monograph **2550** (1956), 1–54.

A table of

$$\pi^{-1}\int_0^y \log |(1+x)/(1-x)|\,dx/x$$

in radians to 7D and in degrees to 5D is given at such an interval that the error in linear interpolation is at most a unit in the last decimal; in fact, $B(y)$ is given for $y = 0(.001).600(.0005).9000(.0001).9940(.00005).99980(.00001)1$. For all these values of y, $B(y^{-1}) = \frac{1}{2}\pi - B(y)$ is also tabulated. These tables are an extended version of 5 figure tables by the same author [Bell System Tech. J. **26** (1947), 870–899; MR **9**, 533]. The table was computed on punched card equipment and seems to have been reproduced photographically from a listing.

Related functions have been tabulated in various connections, and the present tables have been checked using some of them. Spence's integral, $\int_1^y \log u\,du/(u-1)$ has been tabulated by E. O. Powell [Phil. Mag. (7) **34** (1943), 600–607; MR **5**, 110]; K. Mitchell [ibid. **40** (1949), 351–368; MR **10**, 741] has tabulated $\int_0^y -\log|1-y|\,dy/y$. A. Ashour and A. Sabri [Math. Tables Aids Comput. **10** (1956), 57–65] have tabulated $\int_1^y \log|1+t|\,dt/t$ [see also A. Fletcher, Phil. Mag. (7) **35** (1944), 16–17; MR **6**, 50]. *John Todd* (Washington, D.C.).

Roothaan, C. C. J. Study of two-center integrals useful in calculations on molecular structure. IV. The auxiliary functions $C_{\alpha\beta}^{\nu\kappa}(\varrho_\alpha, \varrho_\beta)$ for $\alpha \geqq 0$. J. Chem. Phys. **24** (1956), 947–960.

The computation of hybrid integrals was, in part III [Ruedenberg, Roothaan, and Jaunzemis, same J. **24** (1956), 201–220; MR **17**, 930], made to depend on the computation of

$$\int_1^\infty dx \int_{-1}^1 dy\, e^{-rx-riy} \times (x+y)^a(x-y)^b(1+xy)^c(1-xy)^d(x^2-1)^e(1-y^2)^e,$$

and this integral was supposed to be computed from recurrence relations. Because of the excessive loss of accuracy involved in this process, the author now describes a new method for the computation of the above integral for $a \geqq 0$, and gives tables of auxiliary functions from which the integrals can be computed. *A. Erdélyi.*

See also: Salzer, p. 1203; Klotter, p. 1208.

Mathematical Machines

Czajkowski, Z. Electronic methods of analogue multiplication. I. Electronic Engrg. **28** (1956), 283–287.

Czajkowski, Z. Electronic methods of analogue multiplication. II. Electronic Engrg. **28** (1956), 352–355.

Gavilan, Eduardo. A compact analog computer. Calc. Automat. y Cibernet. **5** (1956), no. 13, 36–41. (Spanish)

Santesmases, J. García. Electronic differential analyzer I. E. — C.S.I.C. Rev. Ci. Apl. **10** (1956), 193–200. (Spanish)

Belevitch, V.; et Storrer, F. Le calcul numérique des fonctions élémentaires dans la machine mathématique IRSIA-FNRS. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 543–578.

Collatz, L.; Meyer, A.; und Wetterling, W. Die Hamburger Integrieranlage "Integromat". Z. Angew. Math. Mech. **36** (1956), 234–235.

Tyroler, J. F.; and Smith, C. L. New computer method for obtaining optimum area. Rev. Sci. Instrum. **27** (1956), 392–394.

Łukaszewicz, L.; and Szeptycki, P. Electronic integrator of differential equations ELI. Zastos. Mat. **2** (1956), 399–415. (Polish. Russian and English summaries)

Soanes, S. V. A new serial digital decoder. Electronic Engrg. **28** (1956), 247–249.

Heggs, P. Principles and application of electronic analogue computers. I, II, III, IV. Electronic Engrg. **28** (1956), 120–122, 168–170, 212–215, 257–259.

Adler, Helmut. Ein Gerät zur Auflösung von Polynomgleichungen. Wiss. Z. Tech. Hochsch. Dresden **5** (1955/56), 1–4.

Perls, Thomas A.; and Sherrard, Emile S. Frequency response of second-order systems with combined Coulomb and viscous damping. J. Res. Nat. Bur. Standards **57** (1956), 45–65.

Curves obtained with an analog computer are presented for the magnification factor versus frequency ratio of second-order systems with combined Coulomb and viscous damping. *From the author's summary.*

See also: Zierler, p. 1174; Moisil, p. 1180.

ASTRONOMY

★ **Milanković, M. Osnovi nebeske mehanike. [Elements of celestial mechanics.] 2d ed.** Naučna Knjiga, Beograd, 1955. 100 pp.

The contents of this introductory book on celestial mechanics, whose first edition appeared in 1947, corresponds to the new curriculum of the Faculty of Mathematics and Natural Science at the Beograd University. The book is largely based on the authors work on the application of vector elements $\mathbf{C}, \mathbf{D}, \tau$ ($\mathbf{C} \cdot \mathbf{D} = 0$) of an orbit to the calculation of perturbations of planetary motions [Acad. Serbe. Bull. Acad. Sci. Math. Nat. A. no. **6** (1939), 1–70; Kanon der Erdbestrahlung und seine Anwendung

auf das Eiszeitenproblem, Acad. Roy. Serbe, Belgrade, 1941; MR **11**, 407]. The exposition is original, concise and clear. The reviewer recommends this text-book highly as an introduction to celestial mechanics.

The chapter headings are as follows. I. Newton's law of gravitation. II. The problem of two bodies in celestial mechanics and the unperturbed heliocentric motions of the planets. III. General integrals of the n-body problem. IV. General theory of planetary perturbations. V. Introduction of vector elements for the calculation of perturbations. VI. Retrospection on the classical theory of perturbations. VII. Secular perturbations.

E. Leimanis (Vancouver, B.C.).

Cox, J. F.; et van den Dungen, F. H. Sur les équations de Poisson, utilisées pour le calcul de la précession et de la nutation. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 245–251.

The paper deals with the kinematical interpretation of the additional terms of the second order, so-called, that arise if the complete equations for the motion in space of the axis of figure of the earth are stated in Poisson's form. *D. Brouwer* (New Haven, Conn.).

Dilgan, H. Sur quelques cas particuliers du problème des deux corps de masses variables. Bull. Tech. Univ. Istanbul **8** (1955), 42–49. (Turkish summary)

Three special cases of the problem indicated in the title are distinguished: (1) The sum μ of the masses of the two bodies varies so slowly that its first and second derivatives are negligible compared to μ. (2) The instantaneous eccentricity is negligible compared to unity. (3) Both of these approximations are valid. In this most restrictive third case the problem is reduced to the solution of a homogeneous linear differential equation of the second order (or, what amounts to the same thing, to a Riccati equation). Consequently, in a number of special cases the problem (with the above approximations) is solvable by means of certain well-known transcendental functions such as Bessel functions. *D. C. Lewis.*

Michailovitsch, Dobrivoje. Eine Anwendung der Störungsgleichungen auf das Radzievsky's Dreikörperproblem. Bull. Soc. Math. Phys. Serbie **7** (1955), 223–228 (1956). (Serbo-Croatian. German summary)

The author treats the Radzievskiĭ case of the three body problem [Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 1309–1311; MR **15**, 653] as a perturbed Kepler motion with given perturbing force. Using the vector elements $\mathbf{C}, \mathbf{D}, \tau$ ($\mathbf{C}\cdot\mathbf{D}=0$) of an orbit, introduced by M. Milanković [Acad. Serbe. Bull. Acad. Sci. Math. Nat. A. no. **6** (1939), 1–70; MR **11**, 407], and the perturbation equations of M. Milanković (ibid.) and B. Popović [Glas Srpske Akad. Nauka **198**. Od. Prirod.-Mat. Nauka (N.S.) **3** (1950), 129–139; MR **14**, 96], the solution of the Radzievskiĭ problem is reduced to the integration of a system of two equations which expresses the variations of the elements $\mathbf{D}$ and τ in the two-body problem. Finally, using the vector elements $\mathbf{C}$ and $\mathbf{G}$ of A. Bilimović [ibid. **191**. Od. Prirod.-Mat. Nauka **96** (1948), 83–115; MR **11**, 466], this last system of equations, which is equivalent to three scalar equations, is replaced by a single vector equation which expresses the variation of the element $\mathbf{G}$ in the two-body problem. *E. Leimanis* (Vancouver, B.C.).

Gazarhi, L. A. On a new integral, algebraic in the velocities, of the generalized three-body problem. Ukrain. Mat. Ž. **8** (1956), 5–11. (Russian)

The author considers the motion of three particles with equal masses, interacting with forces of modulus

$$m_i m_j |Ar_{ij}+Br_{ij}^3| \quad (i, j=1, 2, 3; i\neq j),$$

where r_{ij} denotes the distance between the masses m_i and m_j, and A, B are constants. The existence of a new algebraic integral, independent of the ten classical integrals, is shown. The integral found by Yu. D. Sokolov [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **46** (1945), 95–98; MR **7**, 224] and that obtained by E. Egerváry [ibid. **55** (1947), 793–795; MR **9**, 211] are special cases of this integral. By means of this new integral the order of the system of differential equations of motion is reduced to four. *E. Leimanis* (Vancouver, B.C.).

Rogers, M. H. The propagation and structure of shock waves of varying strength in a self-gravitating gas sphere. Proc. Roy. Soc. London. Ser. A. **235** (1956), 120–136.

Taylor's classical similarity solution [same Proc. **201** (1950), 159–174] for the expanding spherical shock wave in a gas in which a certain quantity of energy has instantaneously been liberated at a point, a solution valid during the initial stages of motion while the shock wave remains strong, has been extended in the present paper to the case when the initial density distribution is spherically-symmetric about that point and varies as the inverse αth power of the distance from it, and the initial pressure distribution is that which results from the self-gravitation of such a mass distribution. The gas is taken to be a perfect gas with constant ratio of specific heats γ. The calculations are performed by a method suggested by recent papers of McVittie [Astrophys. J. **59** (1954), 173–180; MR **16**, 1165] and Sakurai [J. Phys. Soc. Japan **8** (1953), 662–669; MR **15**, 909]. Sufficient analysis is given to make it in principle possible to extend the calculation to other initial density distributions, and to the determination of the first-order deviations from the similarity solution as the shock strength falls, but this work is not carried out in detail.

Complete details of the similarity solutions are given for $\alpha=1.5$ and 2.5, taking successively in each cases $\gamma=1.3$, 1.4 and 5/3. For $\alpha=2.5$ the strength of the shock wave remains constant, as already foreshadowed by Kopal [Astrophys. J. **120** (1954), 159–171; MR **16**, 182]. For $\alpha<2.5$ it decreases with time. For $\alpha=2.5$, practically all the mass which was initially inside the sphere now reached by the shock wave is moving out with the "blast wave" (which is bounded internally by a "contact discontinuity" about half a radius from the centre). For $\alpha=1.5$ only about 30% of that initial mass is in the blast wave, but the latter is confined to a thin shell of thickness about 0.06 radii.

The reviewer finds it difficult to picture in the latter case what is happening to the other 70%. All this gas has been through the blast wave, but apparently on reaching the alleged contact discontinuity something quite new happens to it; precisely what is not stated. The qualitative similarity between the author's results for $\alpha=2.5$ and Taylor's results (which correspond closely enough to $\alpha=0$), and the completely different character of the author's results for the intermediate case $\alpha=1.5$, make the latter set appear a trifle suspicious.

M. J. Lighthill (Manchester).

Hunger, Kurt. Zur Theorie der Wachstumskurven. Z. Astrophys. **39** (1956), 36–60.

In the theory of the formation of absorption lines in

stellar atmospheres, one generally distinguishes lines formed coherently or incoherently under circumstances of absorption or scattering; and one seeks to establish the curve of growth which determines the dependence of the equivalent width of the line as a function of the total effective number of atoms in the line of sight. The curves of growth in case the lines are formed coherently have been derived by M. H. Wrubel [Astrophys. J. **109** (1949), 66–75; **111** (1950), 157–164; **119** (1954), 51–57]. The present paper deals with the problem when the lines are formed incoherently. Curves and tables are presented.

An appendix to the paper discusses the Voigt function

$$H(\alpha, v)=\frac{\alpha}{\pi}\int_{-\infty}^{+\infty}\frac{e^{-y^2}}{\alpha^2+(v-y)^2}\,dy.$$

Series expansions for $\alpha\ll 1$ [cf. D. L. Harris, ibid. **108** (1948), 112–115] and $v\gg 1$ are well-known. These expansions are discussed and a table of the function for $v=0(0.25)4.0(0.5)6(1)10$ and $\log 2\alpha=-0.5,\ 0,\ +0.5$ and 1.0 is given. A nomogram of the function is further provided. *S. Chandrasekhar* (Williams Bay, Wis.).

Bonnor, W. B. The formation of the nebulae. Z. Astrophys. **39** (1956), 143–159.

It is assumed that a condensation in a uniform model of the universe can be represented by a region where the pressure is zero and the density uniform, outside which lies a spherical transition region, this in turn being surrounded by the general distribution of matter in the universe, also assumed to be at zero pressure and of uniform density. The uniform densities are different inside and outside the condensation and both, of course, can vary with the time. It is shown that, for the case when the cosmical constant is assumed to be zero, this model of a condensation could give rise to a nebula but the perturbations required to produce the nebulae by the present time are much greater than would be expected from density-fluctuations in accord with statistical theory. This difficulty is shown to persist for point-source models even with non-zero cosmical constants and it is suggested that such models should be abandoned in favour of those with longer time-scales. *G. C. McVittie.*

Sagrebin, D. W. Die Theorie des regularisierten Geoids. Deutsche Akad. Wiss. Berlin. Veröff. Geodät. Inst. Potsdam no. **9** (1956), 129 pp.

Translation by W. Klemm of Teoriya regulyarizirovannogo geoida, which appeared in Trudy Inst. Teoret. Astr. **1** (1952), 87–222 and was reviewed in MR **17**, 673.

See also: Gross and Ziering, p. 1256.

RELATIVITY

Jessel, Maurice. Sur une extension possible de la notion d'invariance. C. R. Acad. Sci. Paris **240** (1955), 2495–2497.

Invariance with respect to units and coordinate axes having been achieved, the author studies Maxwell's equations independently of the constitutive relations. The equations fall into two independent pairs, for which a symmetrical notation is developed, various potentials being proposed. *F. V. Atkinson* (Canberra).

Arcidiacono, Giuseppe. Sull'importanza del "gruppo base" nel problema della unificazione dei campi fisici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 386–391.

L. Fantappiè has shown [same Rend. (8) **17** (1954), 158–165; MR **17**, 795] that just as the Galilean group can be regarded as the limiting case for $c\to\infty$ of the Lorentz group, so this latter group can be regarded as the limiting case of a "final" group, which cannot itself be regarded as the limit of another group of the same type. The theory of relativity invariant under this final group may be called final relativity, and reduces as a limiting case to special relativity. With a view to obtaining the generalized Maxwell equations for final relativity, the author analyses Maxwell's equations for the electromagnetic field from the fact that they are invariant under the Lorentz group, and points out the well-known result that when the Lorentz group reduces to the Galilean group the equations of the electromagnetic field reduce to those of two independent fields, the electrostatic and the magnetostatic. He intends to follow the same line of argument to find out what happens to the generalised equations when final relativity reduces to special relativity.

A. J. McConnell (Dublin).

Arcidiacono, Giuseppe. Le equazioni di Maxwell generalizzate nella teoria di Relatività finale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 515–519.

Following the argument of the paper reviewed above, the author obtains the generalized equations of Maxwell in the theory of final relativity as defined by Fantappiè [same Rend. (8) **17** (1954), 158–165; MR **17**, 795]. Final relativity reduces to special relativity as a limiting case when a certain constant R tends to ∞. In this limiting case the generalized Maxwell equations divide into two groups of equations quite independent of each other. The first group are the classical Maxwell equations of the electromagnetic field; the second group, which form the equations of a new field of force, can be written in the form $\mathrm{rot}\,\bar{\mathbf{G}}=\mathbf{h}$, $\mathrm{grad}\,\bar{G}_0+\partial\bar{\mathbf{G}}/c\partial t=\mathbf{k}$, $\partial G_0/c\partial t+\mathrm{div}\,\mathbf{G}=0$. The new field is defined by the vectors $\mathbf{G}$ and $\bar{\mathbf{G}}$ (which are to be equal in vacuo) and the scalars G_0, $\bar{G}_0$, and the vectors $\mathbf{h}$ and $\mathbf{k}$ are sources of the field. The properties of this new field have still to be investigated.

A. J. McConnell (Dublin).

Winogradzki, J. Sur le tenseur impulsion-énergie métrique et le théorème de Noether. Cahiers de Phys. no. **67** (1956), 1–5.

The author begins by quoting some results of E. Noether [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1918**, 235–257], who studied the fields obtained from a variational principle in which the action-integral is invariant with respect to a continuous group. Taking Minkowski space with $x_4=ict$, he seeks to deduce the characteristic properties of the special-relativity energy-momentum tensor without any presuppositions regarding its structure. The (non-independent) properties in question are those of having conservative moment, of being conservative, and of being symmetric. Account being taken of the field-equations, he finds that these properties arise, separately, from the assumption that the action-integral is invariant under the group G_6 of rotations (the Lorentz group), the group G_4 of 4-dimensional translations, and the whole group G_{10} of displacements. The nature of the

corresponding problem in general relativity is briefly discussed. *H. S. Ruse* (Leeds).

Peretti, Giuseppe. Caratterizzazione geometrica della relatività ristretta. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 1034–1042.

Using a (3+1)-dimensional formalism, the author studies the motion of a particle moving under any force. He relates the force to the curvature of the particle's world-line. *F. A. E. Pirani* (London).

Wagh, R. V. On the gravitational field of an isolated fluid sphere. J. Univ. Bombay. Sect. A. (N.S.) **24** (1955), no. 38, 1–4.

Wagh, R. V. On some spherically symmetrical models in relativity. J. Univ. Bombay. Sect. A. (N.S.) **24** (1955), no. 38, 5–9.

These two papers are concerned with the solutions of Einstein's equations for metrics of the form

$$ds^2=e^{\nu}dt^2-e^{\mu}(dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\phi^2), \tag{1}$$

where ν and μ are functions of r and t. It is stated that the "usual condition of isotropy" is

$$e^{\mu/2}\left(\frac{1}{2}\mu''-\frac{1}{4}\mu'^2-\frac{1}{2r}\mu'\right)=\phi(r), \tag{2}$$

where $\phi(r)$ is an undetermined function of r, and a prime denotes a derivative with respect to r alone. The condition (2) is not the condition of isotropy determined for (1) by the usual method of introducing co-moving co-ordinates and postulating an isotropic pressure. The author's assumption that e^{μ} is a product of a function of r and a function of t, does in fact imply that $\phi(r)$ is necessarily zero and not, as he seems to believe, that $\phi(r)=0$ is an additional restriction in this case. Thus he arrives correctly at the well-known solutions for uniform models of the universe. His other solutions in which $\phi(r)$ is proportional to r^2 are of doubtful validity in view of the ambiguity as to the meaning and method of derivation of equation (2). *G. C. McVittie* (Urbana, Ill.).

Pham Tan Hoang. Sur le potentiel électromagnétique créé par des particules chargées. C. R. Acad. Sci. Paris **242** (1956), 2103–2106.

Starting from the formula $\phi_\lambda=\sum u_\lambda e/r$ of special relativity for the 4-potential of a set of charges e moving with 4-velocities u^λ, the author imposes a riemannian metric $b_{\alpha\beta}dx^\alpha dx^\beta$ and expands both $b_{\alpha\beta}$ and the charges e in powers of a parameter ε, obtaining for the 4-potential of a set of charges a formula which he compares with formulae obtained from unified field theory by Infeld [Acta Phys. Polon. **10** (1951), 284–293; MR **14**, 97] and Callaway [Phys. Rev. (2) **92** (1953), 1567-1570; MR **15**, 564]. *J. L. Synge* (Dublin).

Maravall Casesnoves, Darío. Attempt at a unitary theory of gravitation and electromagnetism. Rev. Mat. Hisp.-Amer. (4) **15** (1955), 88–114, 165–181. (Spanish)

This is a "non-affine" theory in the KS classification [Kilmister, and Stephenson Nuovo Cimento (9) **11** (1954), supplemento, 91–105, 118–140; MR **16**, 870]. The author defines a parallel displacement identical with that proposed by Novobátzky [Hungar. Acta Phys. **1** (1949), no. 5, 1–6; MR **11**, 746] and develops the geometrical consequences. In the last three pages he suggests a physical interpretation for the equation of geodesics, and proposes field equations. It appears from the latter that the energy-momentum tensor, whose constitution is not explained, will in general be non-symmetric. *F. A. E. Pirani* (London).

Schild, A. On gravitational theories of Whitehead's type. Proc. Roy. Soc. London. Ser. A. **235** (1956), 202–209.

It is shown in this paper that Whitehead's theory of gravitation is only one of many of the same type. All of them would agree as far as the perihelion motion (in the case of a one body problem) and the deflection of light is concerned with the predictions of general relativity theory; all of them would predict a secular acceleration of the center of gravity in the case of a two body problem. *L. Infeld* (Warsaw).

See also: Pham Tan Hoang, p. 1245.

MECHANICS

Fantappiè, Luigi. Deduzione della legge di gravitazione di Newton dalle proprietà del gruppo di Galilei. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 361–367, 458–461.

By assuming that the law of universal gravitation obeys the following conditions: (1) it is invariant under the Galilean group, (2) the functions involved in its expression are regular in the sense of analytic functions at all points including those at infinity, (3) the mutual action of two material points depends only on their positions, and (4) such actions are linearly combined; the author deduces that the only law satisfying these conditions is the Newtonian one of the inverse square. *A. J. McConnell* (Dublin).

★**Blaschke, W.; und Müller, H. R. Ebene Kinematik.** Verlag von R. Oldenbourg, München, 1956. 269 pp. DM 26.80.

Dieses anziehende und klar geschriebene Buch zerfällt in vier Abschnitte. I (pp 9–112) über zwangläufige Bewegungsvorgänge im kleinen behandelt die übliche elementare Kinematik des momentanen Zustandes (Pol, Beschleunigungspol, Bahnkrümmung u.s.w.) und die gebräuchlichen Anwendungen auf die klassischen Mechanismen. Die Methode des Bezugskreuzes in einer von Rast- und Gangebene verschiedene Ebene wird ständig angewandt. Die knappe Darstellung gestattet eine Entwicklung welche Gegenstände wie Kreispunktskurve, Burmestersche Punkte und den Satz von Roberts umfasst. Wir nennen im besonderen: „verwandte Drehpolpläne", die mit dem Bewegungsvorgang verbundene Berührungstransformation, die Anwendung komplexer Zahlen, die Betrachtung der Fernpolstellungen und den Zwillingsinversor von Bereis.

II (pp 113–143) enthält eine wertvolle Zusammenfassung über eingliedrige Bewegungsvorgänge im grossen: Formeln für die Fläche geschlossener Bahnkurven, der Satz von Holditch, die Gesammtkrümmung einer Kurve, Hüllbahnlänge und -fläche, der Sätz von Hennig. Dieses Kapitel ist eine vorzügliche Einführung in die Integralgeometrie.

III (pp 144–176) beschäftigt sich mit zweigliedrigen Bewegungsvorgängen. Es stellt dem Leser wesentlich höhere Anforderungen. Der Ausgangspunkt ist der Begriff

der beiden momentanen Polarachsen und deren Zuordnung, welche „dichtetreu" ist. Bewegungsvorgänge zu vorgegebener Zuordnung und solche mit verschwindender Achsendichte. Betrachtung ausgezeichneter eingliedriger Bewegungsvorgänge in einem zweigliedrigen System.

IV (pp 177–258) ist eine Monographie über die bekannte von Blaschke herrührende kinematische Abbildung, welche durch ihre mannigfäche Berührungen mit vielen geometrischen Sachen so reizvoll ist.

Die Verfasser versprechen im Vorwort ein zweites Buch über räumliche Kinematik. *O. Bottema* (Delft).

Federhofer, Karl. Zur Kinematik des Schleifkurvengetriebes. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. II. **164** (1955), 473–482.

Beim Schleifkurvengetriebe handelt es sich um einen ebenen Mechanismus, der aus zwei Gliedern G_1 und G_2 besteht, die um feste Punkte so rotieren, dasz bezugsweise mit G_1 und mit G_2 verbundenen Kurven sich dauernd berühren. Für G_1 wird eine gewisse Beschleunigung angenommen; Verf. kann dann Geschwindigkeiten und Beschleunigungen an G_2 konstruieren und im besonderen auch dem Krümmungsradius ϱ des Berührungspunktes bei der relativen Bewegung von G_2 bezüglich G_1. Eine Berechnung von ϱ gibt die Formel, welche von Hinkle, Ip und Frame [J. Appl. Mech. **22** (1955), 222–226; MR **16**, 966] auf andere Weise gefunden ist. *O. Bottema.*

Linsman, M. Le graphique des centres instantanés de rotation dans l'étude cinématique des machines. Bull. Soc. Roy. Sci. Liège **25** (1956), 198–208.

Ziegler, Hans. Der symmetrische Kardankreisel unter einem Moment an der Achse des äusseren Rings. Z. Angew. Math. Phys. **7** (1956), 253–256.

Clauser, Emilio. Geometrizzazione della dinamica dei sistemi anolonomi. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 673–682.

In a previous paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13** (1955), 495–501; MR **17**, 795] the author studied the possibility of identifying the trajectories of a holonomic dynamical system of n degrees of freedom q^i with the autoparallels of a so-called non-Riemannian space. A metric is defined in the latter by the quadratic form representing the kinetic energy of the system, but the connection coefficients are not of the usual type, depending, in fact, on directional arguments $\dot{q}^i$ (i.e. the velocity components) as well as on position. The present paper is devoted to an extension of these results to systems which are subjected to non-holonomic constraints of the type $A_{(\nu)i}(q^i)\dot{q}^i=0$ ($\nu=1, \cdots, s$; $i, j=1, \cdots, n$). After writing down the equations of motion according to the classical formalism, the undetermined multipliers are eliminated in a manner similar to a method due to Synge [Philos. Trans. Roy. Soc. London. Ser. A. **226** (1926), 31–106]. In the configuration space of the q^i connection coefficients are then sought such that the equations of motion are identical to the equations of corresponding paths (or autoparallels). These are obviously not unique (again depending on direction), and certain simplifications result from the assumption of more specialised forms of these connection coefficients. In particular a connection similar to that of Weyl is discussed in some detail. *H. Rund* (Durban).

Clauser, Emilio. Geometrizzazione della dinamica dei sistemi a vincoli mobili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 33–39.

In the paper reviewed above the author discusses the possibility of identifying the equations of motion of a non-holonomic dynamical system with n degrees of freedom q^i with the equations of suitably chosen "paths" in the configuration space of the q^i. The latter space is endowed with a metric defined by the quadratic form representing the kinetic energy of the system, and in the above-mentioned paper it is assumed that the s constraints are fixed, i.e. of the form $A_{(\nu)k}(q^i)\dot{q}^k=0$ ($i, k=1, \cdots, n$; $\nu=1, \cdots, s$). In the present paper the author extends his method to the case of moving non-holonomic constraints of the type $A_{(\nu)k}(q^i, t)\dot{q}^k+B_{(\nu)}(q^i, t)=0$, where t is the time. The general method does not differ essentially from that of the previous paper, and it is found that the required connection coefficients consist of sums of three terms: the first term has a structure fundamentally similar to that of the connection coefficients of a Weyl space (but depending implicitly on the velocity components), while the second term is proportional to the velocity components and results from the fact that the constraints are not fixed, and the third term is a consequence of the non-holonomic character of the constraints. *H. Rund* (Durban).

Agostinelli, Cataldo. Nuova forma sintetica delle equazioni del moto di un sistema anolonomo ed esistenza di un integrale lineare nelle velocità lagrangiane. Boll. Un. Mat. Ital. (3) **11** (1956), 1–9.

Non-holonomic dynamical systems with n independent coordinates, in which the $m(<n)$ constraints are linear in the velocities and independent of the time, are treated in essentially the usual way using Lagrange multipliers. Elimination of these multipliers leads to a system of order $2n$, admitting m first integrals linear in the velocities and corresponding to the m constraints. The non-holonomic motions correspond to the solutions of this $2n$th order system for which these m first integrals all vanish. The author then considers necessary and sufficient conditions, couched in geometrical terms, for the existence of additional first integrals linear in the velocities. Many of the equations are written in a somewhat novel vectorial form. *D. C. Lewis.*

Bishop, R. E. D. The behaviour of damped linear systems in steady oscillation. Aero. Quart. **7** (1956), 156–168.

The author proposes to avoid some of the complications in the analysis of frictional damping in dynamic systems by replacing the hypothesis of viscous damping by that of "hysteretic damping". For harmonic oscillations this is proportional to the velocity divided by the frequency, so that its effect is accounted for by replacing the spring constant by a complex number. The physical justification for such an assumption is not discussed, but it greatly simplifies the analysis of the multiple degree of freedom system considered in the paper. *E. Pinney.*

Kondo, Kazuo. On the dynamics of the aeroplane and non-Riemannian geometry. I, II. J. Japan Soc. Aero. Engrg. **2** (1954), 161–166, 193–196.

Whenever two or more types of energies coexist in a system, the non-holonomic Riemannian space of the dynamical equations of Lagrange can be replaced (for purposes of better visualization and manipulation) by a

holonomic, non-Riemannian space with torsion. The replacement consists of substituting the torsion tensor $S_{\alpha\beta\gamma}$ — having a simple law of transformation — in place of the non-holonomic object $\Omega_{\alpha\beta,\gamma}$ which obeys a more complicated law. It is shown in the paper that during the small oscillation of a moving aircraft (subject to mechanical plus aerodynamical forces) the tensorial equations of displaced geodesics assume identical form with those of a uniformly rotating electrical machine (subject to mechanical plus electromagnetic forces) having superimposed small oscillations. *G. Kron.*

Trjitzinsky, W. J. Problèmes dans la théorie des systèmes dynamiques. Acta Math. **95** (1956), 191–289.

In the sense in which it is used in this paper, a dynamical system is a continuous (one-parameter) or discrete (iterates of a homeomorphism) flow defined on a subset of euclidean n-space. The results are largely of a topological nature, although in the last section an ergodic theorem is derived. To a large extent, the results stem from and make use of developments of A. Denjoy [Leçons sur le calcul des coefficients d'une série trigonométrique, Parties I–IV, Gauthier-Villars, Paris, 1941–1949; MR **8**, 260; **11**, 99]. There are generalizations of the notion of recurrence (stability in the sense of Poisson) in the sense that a point does not necessarily return arbitrarily closely to its original position, but instead returns infinitely often within a specified distance. But the main part of the paper is devoted to generalizations of recurrence or stability (Poisson, Liapounoff) to the case where the flow of a set or collection of sets is considered instead of the motion of a point. Typical theorems assert the existence of a residual set having some recurrence property as a consequence of the property occurring on a dense set. There are also generalizations of regional recurrence, the central motions, and minimal center of attraction to the setting indicated.

An ergodic theorem is derived for a space of finite measure with the condition of invariant measure replaced by a condition bounding the measures of the images of a set of positive measure away from zero. The proof follows closely Khintchine's proof of the Birkhoff ergodic theorem. It should be remarked that the same result could be obtained rather easily by showing that the hypotheses imply the existence of an equivalent invariant measure. *G. A. Hedlund* (New Haven, Conn.).

Teodorescu, N. Introduction physico-mathématique à la théorie invariante de la propagation des ondes. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1952), no. 1, 25–51. (Romanian. Russian and French summaries)

The present work can virtually be considered as a unified philosophical exposition of the author's ideas of the invariant theory of wave propagation [cf. N. Teodorescu, Mem. Acad. R. P. Române **23** (1951)]. The waves are formed by the unit-contact elements, while the propagation is of an isolated point perturbation type. In the exposition the author employs the concepts of the Finsler metric, Huygens' principles, Lie's transformation group theory, Hadamard's epicentral wave theory, etc.

The author concludes his study with the following statement.

„··· la description mathématique des phénomènes de propagation par ondes est ramenée a l'étude des équations aux dériviées partielles linéaires, en partant des phénomènes élémentaires, les ondes étant formées d'ondes élémentaires, les corpuscules n'étant plus des points mais des voisinages des centres d'ébranlement contenant un élément de contact sur lequel sont répandus les centres d'ébranlement voisins···.

It seems to the present reviewer that the author is attempting to introduce some kind of a quantization process; however, without any conclusive result as yet. The author does not seem to be familiar with the results obtained by M. Riesz [cf. Acta Math. **81** (1949), 1–223; MR **10**, 713 ; also, the work by Baker and Copson [The mathematical theory of Huygens' Principle, 2nd ed., Oxford, 1950; for a review of the 1st ed. see MR **1**, 315] is not referred to. *K. Bhagwandin* (Oslo).

See also: Consiglio, p. 1210; Vâlcovici, p. 1216 Dilgan, p. 1243; Schubart, p. 1241; Storchi, p. 1217; Birger, p. 1206.

Fluid Mechanics, Acoustics

Šehter, V. M. On single-valuedness of the hydrodynamic mass tensor. Vestnik Leningrad. Univ. **11** (1956), no. 4, 23–26. (Russian)

The author proved [Same Vestnik **9** (1954), no. 11, 99] that in the case of a medium interacting with electromagnetic poles, the mass tensor is determined uniquely up to a constant scale factor. Here an analogous demonstration is given for the case of an ideal fluid. *L. M. Milne-Thomson* (Providence, R.I.).

Gerber, Robert. Sur une classe de solutions des équations du mouvement avec surface libre d'une liquide pesant. C. R. Acad. Sci. Paris **242** (1956), 1260–1262.

The author continues his work on existence theorems for steady, two-dimensional, irrotational gravity flows of an incompressible fluid with a free boundary [see J. Math. Pures Appl. (9) **34** (1955), 185–299, esp. chap. II; MR **17**, 422]. In the previous memoir, the first existence theorem concerned flows in which the profile S of the bottom of the channel was periodic and symmetric, and in which the ordinate of the free line L was assumed to increase and decrease with that of S. In the present paper the author takes up the opposite case, in which the ordinate of L increases while that of S decreases. It is shown that for such solutions, the system of non-linear integro-differential equations used previously can be replaced by a new auxiliary system, under certain restrictions on the solutions. After obtaining suitable a priori bounds for solutions of this auxiliary system, the author establishes the following existence theorem by applying the Schauder-Leray theory of functional equations.

Let γ denote the acceleration of gravity, ψ_0 the total flux, V_0 the maximum speed, L_0 the length of a half period of the curve S, and Δ the change in the ordinate of S over a half period. Then there exists at least one solution to the given problem providing the Froude number $\gamma\psi_0/V_0^3 > a\pi^3/2$, where $a>1$, and if the dimensionless quantities ψ_0/L_0V_0 and $V_0\Delta/\psi_0$ are sufficiently small.

The argument is outlined briefly, but no details of the proof are given. *D. H. Hyers* (Los Angeles, Calif.).

Gheorghiţă, Şt. I. Contribution à l'étude des mouvements dans des milieux poreux. Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 273–356. (Romanian. Russian and French summaries)

The present work (84 pages) is a valuable addition to recent progress in filtration theory. The author analyses

the different aspects of the theory quite carefully, and fills gaps where necessary. Virtually all the notable contributions are mentioned. A translation of this memoir seems to be warranted.

The first chapter is devoted to the derivation of the filtration equations. Certain relations between the velocity and the gradient of the pressure characterise the regions of validity of the Darcy-type approximations. The forces acting on porous bodies are derived and are given in the form of integrals (the appropriate surfaces are subjected to Lyapunov's condition); particular cases are noted. The paradox of d'Alembert is proved to be non-existent in the case of porous bodies.

In the second chapter, filtration relative to a single body as well as to n bodies is considered. The velocity-potential is expanded in terms of the filtration coefficient (considered small) and harmonic functions which satisfy the Neumann and Dirichlet conditions. Applications are made to the cases of circular cylinders, an impermeable medium surrounded by a porous medium, an impermeable medium continued on a porous medium, et al. In the third chapter the preceding analysis is generalised to include the case of viscous fluids.

In the fourth chapter the effects of inhomogeneity are considered. Two cases are noted, viz., 1) two media, one of them surrounded entirely by the other and 2) both media have a common surface and a portion in contact with the fluid. Approximate solutions in terms of harmonic functions are obtained for both cases. Further generalizations are also noted. In the appendix, the author summarises some of the recent theoretical investigations of H. Ya. Baĭciorov [Vestnik Moskov. Univ. **1951** no 10, 23; **1952**, no. 8, 73–87; MR **15**, 71], E. Carafoli [Lucrările Sesiunii generale Ştiinţifice, 1950, Acad. R. P. Române, Bucureşti, 1951, p. 836], et al.

The bibliography contains 71 fundamental references. There are, unfortunately, a large number of printing errors in the paper; some of them are noted in list of errata. *K. Bhagwandin* (Oslo).

Tomotika, S.; and Fujikawa, H. The influence of vortices upon the drag experienced by an elliptic cylinder moving through an inviscid liquid. Mem. Coll. Sci. Univ. Kyoto. Ser. A. **27** (1955), 227–241.

The drag due to the existence of a pair of vortices behind the cylinder is calculated.

L. M. Milne-Thomson (Providence, R.I.).

Bader, W. Zum Reibungseinfluss auf die Düsenströmung. Z. Angew. Math. Mech. **36** (1956), 233–234.

Nikitin, A. K. On flow of a viscous incompressible fluid about a sphere. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **32** (1955), no. 4, 43–53. (Russian)

The authors writes $-\chi+\partial\varphi/\partial r$, $(1/r)\partial\varphi/\partial\theta$ for the radial and transverse velocity components and sets up two partial differential equations for χ and φ, one equation being non-linear. Starting with a first approximation $\chi_1=0$, $\varphi_1=U(r+a^3/2r^2)\cos\theta$, a second approximation is found, and the resistance calculated. This turns out to be the same as the well-known formula of Stokes.

L. M. Milne-Thomson (Providence, R.I.).

Faĭnzil'ber, A. M. The energy form of the equations of motion of a viscous fluid and the integral of slightly vortical flows. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 793–796. (Russian)

The two-dimensional equations of motion for viscous incompressible fluids are transformed into an equation devoid of viscosity (the equation is non-linear). The analysis is based on the introduction of the function

$$H=\frac{V^2}{2}+\frac{p}{\varrho}+U+a\omega,$$

a is a constant, $V=[u_x^2+u_y^2]^{\frac{1}{2}}$, U=inertial potential, ϱ is the density, p is the pressure, and $\omega=\partial U_y/\partial x-\partial U_x/\partial y$ (the vortex). After some calculations the author establishes the result $dH/dt=\nu(\nabla^2H-\omega^2)$. For slightly vortical flows the ω term can be neglected; by virtue of which the author obtains the expression H=const. As an example, the author obtains a very simple expression for the velocity distribution in a specific boundary layer.

K. Bhagwandin (Oslo).

Moisil, Gr. C. La méthode des fonctions analytiques dans l'hydrodynamique des liquides visqueux. Com. Acad. R. P. Romîne **5** (1955), 1411–1419. (Romanian. Russian and French summaries)

The author applies the complex-function methods of Kolosov [Z. Math. Phys. **62** (1914), 384–409] and Mushelišvili [Some fundamental problems of the mathematical theory of elasticity, 4th ed., Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **16**, 1067] to the slow motion of viscous incompressible fluids. The flow field can be determined explicitly in terms of two holomorphic functions. A related procedure can be found in a recent paper of Imai [Proc. Roy. Soc. London. Ser. A. **224** (1954), 141–160; MR **15**, 998]. *K. Bhagwandin* (Oslo).

Kanwal, R. P. Generalized periodic singular points with applications to flow problems. Bull. Calcutta Math. Soc. **47** (1955), 9–22.

The author discusses the vibration of a sphere, a circular cylinder and an elliptic cylinder in a viscous incompressible fluid, using the linearized equations of motion. *J. B. Serrin* (Minneapolis, Minn.).

Geis, Theo. Elementarer Existenzbeweis für die Grenzschichtströmung an einer Klasse rotierender Körper. Z. Angew. Math. Mech. **36** (1956), 222–229. (English, French and Russian summaries)

For the boundary-layer flow on certain bodies of revolution rotating about their axes, the solution is required of the equations

$$F'''+cGF''+2G'(1-F')=0$$
$$G'''+cGG''-G'^2+(1-F')^2=0$$

with the boundary conditions $F'(0)=G(0)=G'(0)=0$, $F'(\infty)=1$, $G'(\infty)=0$. The existence of at least one solution of the boundary-value problem is proved, by elementary methods, by generalization of a method applied first by Iglisch [Z. Angew. Math. Mech. **33** (1953), 143–147; MR **15**, 262]. (From Author's summary.) *D. W. Dunn.*

Geis, Theo. "Ähnliche" dreidimensionale Grenzschichten. J. Rational Mech. Anal. **5** (1956), 643–686.

The equations for incompressible steady boundary-layer flow over a smooth curved wall are studied, using a curvilinear orthogonal coordinate system with metrics g_1 and g_2 in the wall ($z=0$). It is proposed to find under what conditions similar solutions exist; i.e., the profiles of the two velocity components parallel to the wall are given by two universal functions of a single similarity variable. To do this the boundary-layer equations are

first expressed in terms of F. K. Moore's stream functions [NACA Tech. Note no. 2279 (1951); MR 12, 872], appropriately generalized. Then a similarity variable is assumed to exist and the boundary-layer equations are transformed accordingly, becoming thereby a pair of simultaneous ordinary differential equations. A similar solution exists only if the coefficients in these equations are constants, and this (necessary) condition leads to a set of partial differential equations which must be satisfied by the external flow and the metrics g_1 and g_2. It is pointed out that the existence of similar solutions is not independent of the choice of coordinates.

Solution of this set of equations is undertaken only for certain especially simple cases. These lead to (a) the well-known Falkner-Skan cases of plane flow, (b) the corresponding axisymmetric flows, (c) the cases wherein the potential flow is source or sink flow, (d) flow due to rotation of a body of revolution about its axis in fluid otherwise at rest and the analogous case of circular flow about a body of revolution (e) flow with parabolic potential streamlines over a plane or wedge [cf. H. Loos, J. Aero. Sci. 22 (1955), 35–40; MR 16, 639], (f) flow with spiral potential streamlines, (g) a family including Howarth's general stagnation-point flow [Phil. Mag. (7) 42 (1951), 1433–1440; MR 13, 505]. *W. R. Sears.*

Rheinboldt, W. Zur äusseren Randbedingung bei den Grenzschichtgleichungen. Z. Angew. Math. Mech. **36** (1956), 153–154.

For the problem of plane, laminar, stationary, and incompressible boundary layer flow around a body, let $\psi(x, y)$ be the stream function, satisfying the usual boundary layer equation. It is customary to specify an initial condition on ψ_y at $x=0$ and boundary conditions on ψ and ψ_y at the wall $y=0$ and on ψ_y at the outer edge $y\to\infty$. The author has shown previously [H. Görtler and W. Tollmien (ed.), 50 Jahre Grenzschichtforschung, Vieweg, Braunschweig, 1955, pp. 328–333] that under certain assumptions the outer boundary condition is superfluous and is automatically satisfied if the initial condition is properly specified. One of the assumptions made is that $\psi(x, y)$ is analytic with respect to x. In the present paper, the proof is carried out with this hypothesis replaced by the weaker one that, for sufficiently large y, $\psi(x, y)$ can be represented for small x by an asymptotic expansion in in powers of x. *D. W. Dunn* (Baltimore, Md.).

★ Townsend, A. A. The structure of turbulent shear flow. Cambridge University Press, New York, 1956. xii+315 pp. $7.50.

The book under review will be of interest to all who are working on the problem of turbulent fluid motion. Since there is as yet no complete mathematical theory of turbulence, present knowledge of the subject is based mainly on experimental studies, many of them carried out during the last ten years. The results are scattered through the literature and are not presented in terms of a single consistent view of the problem. The author, who has himself made important contributions, gives in the present book a systematic discussion of all kinds of turbulent motion on the basis of this experimental work. The experimental results are carefully examined with the aid of existing theoretical knowledge, and interpreted from a unified point of view.

The book begins with two introductory chapters on the equations of motion for turbulent shear flow and the general theoretical concepts required to describe the character of the turbulence. Chapter 3 considers flows with essentially constant mean velocity, and Chapter 4 the uniform distortion of homogeneous turbulence. Chapters 5 and 6 present the basic ideas regarding turbulent shear flow, in the simplest case when there are no solid boundaries. The author discusses here his very interesting concept of the equilibrium of the large eddies, which he suggests may ultimately lead to a complete theory of the mean motion. In Chapters 7 through 12, detailed discussions are given of various special types of shear flow, including wakes, jets, the flow in pipes and channels, boundary-layer flow, and the flow between rotating cylinders.

A few detailed comments may be in order. In particular, it might have been advisable to elaborate further on the significance of the three conditions for the existence of a layer in which the logarithmic velocity distribution holds (see equations (11.3.1) and (11.3.2)). These restrictions seem to imply that the theoretical results subsequently developed are not valid for boundary layers in adverse pressure gradients sufficiently close to separation. Since the limiting case of zero wall stress is referred to on several occasions, the author's viewpoint is not entirely clear. In a first reading of the book, very few errors or misprints were noticed. Most are fairly obvious, with the exception of equation (11.3.3), p. 264, where it appears that the last term is incorrect and should be replaced by

$$(1/2K^2)(d\tau_0/dx)\{[\log(\tau_0^{\frac{1}{2}}y/\nu)+A-1]^2+1\}y.$$

Although the author's discussions of the physical processes occurring in turbulent motion will be very valuable to the experienced research worker, they may make rather difficult reading for one approaching the subject for the first time. Little advanced mathematics is used, but the book would seem to require a considerable background in fluid mechanics. Another general impression is that the historical perspective of the book is somewhat limited, since as a rule complete references are given only for experimental work and not for theoretical ideas. This policy extends to the author's own ideas, some of which are presented here for the first time.

In conclusion, it should be emphasized that the above criticisms are relatively minor, and do not detract from the overall value of the book. The author has succeeded admirably in his task, and the result is a most welcome survey of present knowledge of turbulent shear flow. It appears to be the best single source of information regarding the physical facts that must be considered in any comprehensive mathematical theory of turbulence. The reviewer recommends the book to all who are interested in the subject. *D. W. Dunn* (Baltimore, Md.).

Lighthill, M. J. The image system of a vortex element in a rigid sphere. Proc. Cambridge Philos. Soc. **52** (1956), 317–321.

It is shown that a vortex element has the following image system. (i) A transverse vortex element at the inverse point equal in strength to minus the transverse resultant of the given vortex element divided by the ratio of its distance from the centre to the radius of the sphere. (ii) A radial vortex element at the inverse point, equal in strength to plus the radial resultant of the given vortex element divided by the same ratio. (iii) A uniform line vortex stretching from the inverse point to the centre of the sphere, equal and opposite in total strength to the radial vortex element at the inverse point.

Strength is defined as the product of the length of the element and the circulation round it.

L. M. Milne-Thomson (Providence, R.I.).

Naylor, Derek. Unsteady simple waves in gases. J. Rational Mech. Anal. **5** (1956), 687–714.

The author looks for isentropic unsteady potential plane flows of a polytropic gas for which the velocity vector has the form $\bar{q}(\lambda, t)$ where $\lambda=\lambda(\bar{r}, t)$ (simple waves). He notes that the Legendre transform Φ of the velocity potential ϕ is a function of λ and t alone, and that the curves λ=const, t=const are straight lines, the λ-lines. Application of the Legendre transformation to the basic differential equation for ϕ then leads to three partial differential equations for $q(\lambda, t)$, $\theta(\lambda, t)$, and $\Phi(\lambda, t)$. The first equation does not contain $\Phi(\lambda, t)$ and identifying the parameter λ with q it reduces to a parabolic Monge-Ampere equation for θ. The solution is then obtained in terms of two arbitrary functions, $g(\lambda)$ and $h(\lambda)$. The second equation is second order linear in $\Phi(\lambda, t)$ and using the previous results $\Phi(\lambda, t)$ is obtained as a cubic in t and with four arbitrary functions of λ. By suitably selecting one or more of these functions the solution of the third equation can be reduced to solving several simultaneous ordinary differential equations. This is carried out in three different ways to obtain three different unsteady flow fields. Attention is devoted to the behavior of the envelope of the λ-lines, and in particular it is shown that there are no unsteady (non-radial) centered simple waves.

R. H. Wasserman (Ann Arbor, Mich.).

Jobson, D. A. On the flow of a compressible fluid through orifices. Proc. Inst. Mech. Engrs. **169** (1955), 767–776.

By making certain basic assumptions, the author has determined a theoretical expression for the contraction coefficient, C, appropriate to an orifice when transmitting a compressible fluid, either above or below the critical pressure ratio, provided that the corresponding value for incompressible flow, C_i, be known. *Author's summary.*

Ferrari, Carlo. Sulla determinazione della forma esterna del condotto di rotazione di minima resistenza, per determinate condizioni poste alla linea meridiana, in corrente supersonica linearizzata. Mem. Accad. Sci. Torino. Cl. Fis. Mat. Nat. (3) **1** (1955), 83–138.

The problem treated by the author is as follows. An axisymmetrical duct with an open, sharp-lipped front orifice, has a given internal meridian section. The area (or, alternatively, the volume) between the internal and external meridian sections is given. It is required to find the shape of the external meridian section for which the external drag given by the linearized theory of steady supersonic axisymmetrical flow will be a minimum. Two methods of solution are given and applied to various examples. The interesting result emerges that in every case one gets to a close approximation the same optimum shape as one would get by a theoretical approach in which the simple Ackeret (two-dimensional) pressure formula was applied — even though in the cases under consideration the actual pressure distribution departs substantially from that predicted by the Ackeret formula.

M. J. Lighthill (Manchester).

Miles, John W. A first-order formulation of the unsteady supersonic flow problem for finite wings. J. Aero. Sci. **23** (1956), 578–582.

Quand la fréquence réduite k est petite, l'équation des ondes cylindriques amorties qui governe les mouvements vibratoires d'une aile mince en écoulement supersonique peut être réduite, en négligeant les termes en k^2 à l'équation des ondes cylindriques. Si l'aile a un bord de fuite entièrement supersonique, les pressions sur l'aile s'obtiennent en résolvant un problème aux limites qui est exactement celui rencontré dans l'étude des mouvements stationnaires. Si l aile présente un bord de fuite subsonique il faut de plus considérer un autre problème, d'ailleurs analogue, car la condition à écrire sur le sillage est linéaire en k. Par cette méthode, on obtient ainsi la valeur de la solution à k^2 près. A titre d'exemple, la cas d'une aile trapézoïdale est traité en détail. *P. Germain* (Paris).

Nocilla, Silvio. Campi di moto transonici attorno a profili alari simmetrici, senza incidenza, con numero di Mach 1. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 46–62.

This is a further instalment of the author's work, extending the utility of the theory of Tomotika and Tamada [Quart. Appl. Math. **9** (1951), 129–147; MR **13**, 179], which was begun in two earlier papers [same Atti **89** (1954–55), 296–322; Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 55–61; MR **17**, 1023]. In addition to the functions mentioned in the reviews of those papers, two new functions $\psi_{-3/2}$ and $\psi_{-5/2}$ are introduced by differentiations of $\psi_{-1/2}$ with respect to β. The author then uses these to construct a flow in the hodograph plane corresponding to symmetrical transonic flow about an aerofoil, up to the sonic line and beyond it to the so-called "limiting characteristic". Some considerable choice of aerofoil shape so obtained is available. The work seems a real advance on previous analytical solutions to problems of this type, which had been forced to use the less accurate Tricomi equations of motion.

M. J. Lighthill (Manchester).

Cabannes, Henri. Lois de la réflexion des ondes de choc dans les écoulements plans non stationnaires. O.N.E.R.A. Publ. no. **80** (1955), 29 pp.

A general detailed summary is presented of regular and Mach reflection of a uniform plane shock in a perfect gas from a straight convex corner or wedge. Particular attention is given to the cases of limiting Mach numbers and wedges angles. For those cases for which several solutions are found a choice is made on continuity, minimum entropy change, or experimental considerations. On the basis of these choices transition Mach numbers are found (for constant wedge angle) at which the reflection discontinously changes character from regular to Mach reflection and an "instability" of the flow is said to occur. No mention is made of the fact that for a real fluid this analysis is not valid in this range because of predominating viscous and heat conduction effects. Several remarks are made on the flow behind the reflected waves taking account of the disturbance propagated from the corner of the wall and of the pseudo-stationary character of this flow. *P. Chiarulli* (Washington, D.C.).

Yakimov, Yu. L. On asymptotic solutions of the equations of one-dimensional unsteady motion of an ideal gas and on asymptotic laws of attenuation of shock waves. Prikl. Mat. Meh. **19** (1955), 681–692. (Russian)

The unsteady motion discussed has spherical symmetry. If r is the radial distance, t is the time, u and a are the fluid velocity and velocity of sound respectively, charac-

teristic variables ξ_1, ξ_2 are defined by the relations

$$\mu_{10}d\xi_1=dr-(u+a)dt,$$

$$\mu_{20}d\xi_2=dr-(u-a)dt,$$

where μ_{10}, μ_{20} are integrating factors.

The equations of motion in characteristic form are solved in series of terms $\ln^t r/r^k$ with coefficients depending on ξ_1 or ξ_2, and asymptotic solutions for large r are deduced.

The results are used to calculate the asymptotic behaviour of spherical blast waves. It is shown that the main N wave is followed by a series of waves of decreasing amplitude. *M. Holt* (Providence, R.I.).

Berman, Julian H. Lift and moment coefficients for an oscillating rectangular wing-aileron configuration in supersonic flow. NACA Tech. Note no. **3644** (1956), 46 pp.

Power series approximations to the section coefficients for the title problem are determined through the fifth power of the reduced frequency and the results applied to a typical flutter problem. The reviewer notes that it is possible (albeit not necessarily preferable) to obtain the required spanwise integrals of the section coefficients for polynomial spanwise deflections without expanding in the frequency parameter [Miles, Unsteady supersonic flow, Air Research and Development Command, Baltimore, Md., 1955, sec. 7.6; MR **17**, 426; cf. the treatment of the title problem for rigid wing motions, U.S. Naval Ordnance Test Station, Inyokern, California, Navord Rep. **1185** (1949); **1292** (1951)]. *J. W. Miles.*

Skinner, George Tolmie. Time correlator for problems in aerodynamics. NACA Tech. Note no. **3682** (1956), 32 pp.

Garabedian, P. R. The mathematical theory of three-dimensional cavities and jets. Bull. Amer. Math. Soc. 62 (1956), 219–235.

The theory of axially-symetric steady flows with free streamlines has been extensively studied in recent years, but most discussions have centered on questions of existence and uniqueness, and little has been learned about the explicit solution of specific problems. The paper under review describes a systematic procedure for numerical solution of the free boundary problem which was developed to fill the gap. The author's method, which he illustrates by application to the Riabouchinsky flow past a circular disk, consists in the determination of the stream-function ψ by linear interpolation. Specifically, certain particular solutions $p_1, \cdots, p_m$ of the streamfunction equation

$$\psi_{xx}+\psi_{yy}-y^{-1}\psi_y=0$$

are selected, and one sets $\psi=\sum_{i=1}^{m}\lambda_i p_i$, where λ_i are constants to be determined by appropriate boundary conditions. Here the difficulty of not knowing ahead of time where the free boundary will be is overcome by setting a certain modified boundary condition on a curve which, one hopes, is a good approximation to the true free boundary — (this curve may, for example, be the free boundary for the corresponding plane problem). The λ_i will then be determined by the satisfaction of the boundary condition at m predetermined points on the approximate boundary. In the Riabouchinsky flow already mentioned, this procedure is carried through in detail (with $m=20$), leading to a quite accurate determination of the true free boundary and the drag coefficient. *J. B. Serrin.*

Curle, N. Unsteady two-dimensional flows with free boundaries. I. General theory. Proc. Roy. Soc. London. Ser. A. **235** (1956), 375–381.

This paper deals with unsteady free-boundary flows in the stages before they have come to steady-state conditions. Typical of the situations considered is the problem of a plane jet issuing from a slit in a straight infinite wall. The flow is investigated a long, but finite, time after the slit is opened, so that the nose of the jet has passed far downstream and the free boundary has come near its steady-state position (except near the nose, of course). The velocity potential is expanded in the form

$$\phi(x, y, t)=\phi_0(x, y)+\epsilon^{-\lambda t}\phi_1(x, y)+\cdots,$$

where ϕ_0 is the known steady-state solution and ϕ_1 is a perturbation potential to be determined. Appropriate boundary conditions for ϕ_1 are established, but the determination of ϕ_1 is deferred to the paper reviewed below. *J. B. Serrin* (Minneapolis, Minn.).

Curle, N. Unsteady two-dimensional flows with free boundaries. II. The incompressible inviscid jet. Proc. Roy. Soc. London. Ser. A. **235** (1956), 382–395.

In this paper the author determines the potential ϕ_1 for the problem of a jet issuing from a slit in a straight wall [see the preceding review]. The solution is handled by Fourier methods, although the detailed calculations required the aid of computation services. The resulting jet shape is shown at several stages of its development. Also discussed is the initial shape of the jet immediately after the slit is opened. *J. B. Serrin* (Minneapolis, Minn.).

Legendre, Robert. Influence de l'émission d'un jet au bord de fuite d'un profil sur l'écoulement autour de ce profil. C. R. Acad. Sci. Paris **242** (1956), 2438–2440.

Let a thin jet of fluid be emitted at the rear edge of a profile in a uniform stream of incompressible fluid. Assuming that the resulting motion is steady and irrotational both in the jet and exterior to it, the author shows that in computing the flow outside the jet one may consider the jet to be a vortex line, with a discontinuity of velocity propotional to the local curvative. This condition serves to determine the location of the jet.

J. B. Serrin (Minneapolis, Minn.).

Malavard, Lucien. Sur une théorie linéaire du soufflage au bord de fuite d'un profil d'aile. C. R. Acad. Sci. Paris **242** (1956), 2440–2442.

The flow model discussed in the paper reviewed above is here treated on the basis of the linearized theory. The pertubation potential ϕ_1 is shown to be the solution of a certain boundary value problem in the upper half plane, the boundary conditions being of mixed type.

J. B. Serrin (Minneapolis, Minn.).

Siestrunck, Raymond; et Bernard, Jean-J. Sur une méthode de détermination numérique des potentiels de perturbation dans la théorie linéarisée du soufflage au bord de fuite des profils d'ailes. C. R. Acad. Sci. Paris **242** (1956), 2443–2445.

The boundary value problem formulated in the preceding paper is here solved for a straight profile, the solution being effected by a Fourier series of which the first terms are calculated. *J. B. Serrin.*

Polyahov, N. N. The theory of a grid of plates according to N. E. Žukovskiĭ and its generalization to the case of an arbitrary profile. Vestnik Leningrad. Univ. **11** (1956), no. 1, 125–144. (Russian)

Let $V^*(\zeta)=u-iv$ be the easily constructed conjugate complex velocity for steady incompressible flow without circulation in the $\zeta=\xi+i\eta$ plane at angle of attack β about a grid G of plates $|\xi|\leq a$, $\eta=int$, $n=0, \pm1, \pm2, \cdots$. N. E. Žukovskiĭ has observed that for properly chosen constant of integration the complex potential $z=x+iy=\int V^*(\zeta)d\zeta$ has the property $z(\zeta+it)=z(\zeta)+ite^{-i\beta}$. Accordingly $z=z(\zeta)$ maps G onto another grid of plates with spacing t and stagger β. The author develops a theory for flows through grids of more general profiles by appealing to various analogs of methods used to map an isolated plate or circle onto an isolated profile. For example, $z_1=\zeta_0+\sum_0^\infty (a_n+ib_n)/\zeta_1^{-n}$, where $\zeta_1(\zeta_0)=th(\pi\zeta_0/te^{i\beta})=|\zeta_1|e^{i\theta}$ maps a grid of ovals $|\zeta_1(\zeta_0)|=A$ (or $\zeta_0=\zeta_0(\theta, \beta, A)$) with spacing t and stagger β onto a grid of plates in the z plane with the same spacing and stagger if a_n and b_n are determined by $\eta_0(\theta, \beta, A)+\sum_0^\infty (b_n\cos n\theta-a_n\sin n\theta)A^{-n}=0$. More generally, a grid of ovals $|\zeta_1|=A$ is mapped onto a grid of thin profiles by

$$\tilde{z}=\tilde{x}+i\tilde{y}=z_1+\sum (A_n+iB_n)/\zeta_1^{-n}$$

for the z_1 defined above. The author describes an iterative procedure to determine A_n, B_n approximately for prescribed grid profiles. He also sketches a method to extend this to the case of thick, strongly curved profiles. Once a grid of plates has been mapped onto approximately the desired grid, the corresponding velocity distribution for flow with circulation can be found by straightforward calculations. *J. H. Giese* (Aberdeen, Md.).

Lauwerier, Hans A. Alcuni aspetti della matematica del Mare del Nord. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 116–124.

This is the translation of a lecture (given originally in English) surveying the mathematical work, principally done in the Amsterdam Mathematical Centre, on the motions induced in a sea similar to the North Sea by the combined action of wind stress, bottom friction and Coriolis force. The basic equations are discussed, and then brief accounts are given (with full references) of the active work which has been pursued in this field since the disastrous floods of 1953. *M. J. Lighthill.*

Philipzik, W. Zur hydrodynamischen Theorie der Schmiermittelreibung. Z. Angew. Math. Mech. **36** (1956), 51–60. (English, French and Russian summaries)

From the ordinary Reynolds' equation the author derives the equation $\Delta u-f(x)=g(x)$, $f(x)$ and $g(x)$ are combinations of trigonometric functions; $u=0$ on the boundary. The author solves this equation for the case of a semi-enclosed cylindrical bearing by means of relaxational methods (the solution of a system of 24 equations seemed to yield sufficiently accurate results). The axial pressure distribution is examined, and the results are displayed in several graphs. An elementary minimum theorem is also stated. The author does not seem to be familiar with the work of G. B. Jeffery [Proc. London Math. Soc. (2) **14** (1915), 327–338]. *K. Bhagwandin* (Oslo).

Černov, L. A. Correlation of fluctuations of amplitude and phase in propagation of a wave in a medium with random inhomogeneities. Akust. Ž. **1** (1955), 89–95. (Russian)

L'auteur calcule le coefficient de corrélation des fluctuations de l'amplitude et de la phase au point de réception. On montre que dans le cas des hétérogénéités assez fortes l'autocorrélation des amplitudes (ou phases) s'étendent aussi loin que la corrélation entre les fluctuations de coefficient de réfraction du milieu.

Tous ces résultats ont été obtenus en supposant que le milieu est isotrope, que les variations de l'hétérogénéité sont tout à fait arbitraire du coefficient de réfraction, sont petites par rapport à sa valeur moyenne, mais leurs valeurs sont grandes par rapport à la longueur d'onde. *M. Kiveliovitch* (Paris).

Pachner, Jaroslav. On the dependence of directivity patterns on the distance from the emitter. J. Acoust. Soc. Amer. **28** (1956), 86–90.

A method is developed for determining the directivity pattern in an arbitrary distance from the emitter from measured instantaneous values of the field in other distance. Numerical computation is facilitated by the tables included. *C. J. Bouwkamp* (Eindhoven).

Pachner, Jaroslav. Investigation of scalar wave fields by means of instantaneous directivity patterns. J. Acoust. Soc. Amer. **28** (1956), 90–92.

The author's abstract. A method is proposed for determining the traveling and standing components of an arbitrary scalar wave field from the measured instantaneous values of the field on the surface of two spheres surrounding the emitter. *C. J. Bouwkamp.*

Mullins, W. W. Two-dimensional motion of idealized grain boundaries. J. Appl. Phys. **27** (1956), 900–904.

To represent ideal grain boundary motion in two dimensions, a rule of motion of plane curves is considered whereby any given point of a curve moves toward its center of curvature with a speed that is proportional to the curvature. A general theorem is deduced concerning the change of area enclosed by such a curve. Three families of curves are found that obey the curvature rule of motion while undergoing the shape preserving transformations of uniform magnification, translation, and rotation respectively. *From the author's summary.*

Kidder, Ray E. Flow of immiscible fluids in porous media: exact solution of a free boundary problem. J. Appl. Phys. **27** (1956), 867–869.

A mathematically exact solution is obtained of a two-dimensional free boundary problem arising in the flow of two immiscible fluids in porous media. The particular problem dealt with concerns the "fingering" of water toward a line of equally-spaced oil wells that penetrate a thin, dipping sand. The solution is obtained by use of the hodograph plane and complex variable theory.

Author's summary.

Eckart, Carl; and Ferris, Horace G. Equations of motion of the ocean and atmosphere. Rev. Mod. Phys. **28** (1956), 48–52.

La variation de la densité et de la température de l'atmosphère en fonction de l'altitude ont fait l'objet d'un grand nombre de travaux. Par suite des hypothèses simplificatrices introduites par les différents auteurs on obtient des résultats tout à fait différents qui ne sont même pas comparables. L'auteur a essayé dans le présent travail d'introduire une notation unique et de comparer les résultats obtenus par différents auteurs. *M. Kiveliovitch.*

See also: Davis, p. 1214; Rogers, p. 1243.

Elasticity, Plasticity

★ **Babuška, Ivo; Rektorys, Karel; and Vyčichlo, František. Matematická theorie rovineé pružnosti. [The mathematical theory of plane elasticity.]** Naklad. Československ Akad. Věd, Prague, 1955. 527 pp.

The book is devoted to the application of the theory of functions of a complex variable to solving plane problems of the classical mathematical theory of elasticity (for static problems without the effect of body forces). From the mathematical point of view it deals with the special method of solving a biharmonic equation for given boundary conditions. The book gives and further develops some of the results of N. I. Muskelishvili and collaborators. An original contribution is the axiomatic construction of the fundamentals of plane elasticity, the accuracy and generality of the mathematical procedures and some new numerical methods of solution. However, the book does not exhaust the whole subject; for example, it does not deal with the mixed boundary value problem or estimations of the errors of numerical solutions. The authors plan to deal with these questions and other methods of solving the plane problem of the mathematical theory of elasticity in further volumes.

The whole book is written in a modern mathematical style of the type: definition - theorem - proof, supplemented by brief and comprehensive explanations.

The book is divided into four chapters and a mathematical supplement.

The first chapter is devoted to fundamental concepts. These basic concepts of the plane problem are introduced directly by definition in the plane, not — as is usual in the literature — by specialisation from the spatial problem to plane stress and plane deformation. Thus it directly gives definitions of a plane body, vector and tensor, stress in a plane, deformation etc.; the concept of equilibrium and Hooke's law is also given by definition. The basic concepts are discussed and a derivation given of the fundamental equations for the static case. The authors avoided the procedure usual up to now in the mathematical theory of elasticity, when mathematical and physical considerations alternate. Thus basically they give the axiomatic construction of the fundamentals of plane elasticity.

The second chapter gives the formulation of the problem of plane elasticity. The complex functions of stress are introduced by a procedure similar to that used by Muskelishvili and their characteristics are studied in simply and multiply connected, limited and infinite bodies. The formulation and proof is given of the uniqueness of the solution of the first problem (the components of external stress or that of the principal vector are given on the boundary) and of the second problem of elasticity (the components of displacement given on the boundary).

The third chapter is devoted to methods for solving the problems formulated in the second chapter, by means of complex functions of stress. Only two of the greater number of possible formulations of the problem are dealt with — those using the integral equations of Muskelishvili and Lauricella-Shermann. Greater attention is paid to the equation of Lauricella-Shermann which is used both for proving the existence and uniqueness of the solution and for the general solution of the first problem for a circle, half-plane and the exterior of a circle. Their properties are also used for a new formulation of Saint-Venant's principle. Some new numerical methods for solving the problem of plane elasticity are proposed, consisting in a combination of the method of the functions of a complex variable and the method of the minimum of a quadratic functional. This method is used for solving the second boundary value problem for an annulus and for calculating the stress in a crane hook. At the end of the chapter the possibility is shown of transforming the Lauricella-Shermann equation for multiply connected bodies into a system of integral equations and the advantages of this procedure for numerical calculation is proved.

The fourth chapter gives a solution of the plane problem of elasticity by means of the method of conformal mapping for bodies which can be conformally mapped onto a circle. The corresponding integral equations are given and an approximative solution carried out for a square compressed by two (isolated) point forces.

The mathematical supplement deals briefly with some selected topics of the theory of the functions of a complex variable and also that of integral equations the results of which are used in the preceding chapters.

František Kroupa (Praha).

Prager, W. On conjugate states of plane strain. J. Mech. Phys. Solids **4** (1956), 167–171.

A principle of duality is established for pairs of associated problems involving plane strain in incompressible materials, the relation (not restricted to be linear) between the stresses and strains in the materials for the problems being reciprocal. This result includes previous work by R. Hill [same J. **4** (1955), 1–9; MR **17**, 316] for elastic solids as a special case. In the present paper, the duality for Maxwell and Voigt materials is considered. The significance of this work is that the known solution of one problem may be used to furnish the solution of the associated problem. *H. G. Hopkins* (Sevenoaks).

Wegner, Udo. Bestimmung der Randverschiebungen bei ebenen Spannungsproblemen. Z. Angew. Math. Mech. **36** (1956), 192–198. (English, French and Russian summaries)

The author presents a method for the approximate determination of the displacements at points on the boundary Γ of a disc-shaped body B, when the normal components of the stresses are prescribed on Γ. The exact solution to the problem may be expressed in terms of a biharmonic Airy stress function F which satisfies on Γ the condition $F=f$, $\partial F/\partial n=g$. In the author's treatment of the problem the boundary values of the displacements are expressed in terms of the harmonic function Φ which makes the expression

$$\tfrac{1}{2}\iint_B \Phi^2 dxdy+\int_\Gamma \left(f\frac{\partial\Phi}{\partial n}-g\Phi\right)ds$$

an extremum. For the exact solution $\Phi=\Delta F$. In his method, the author is not required to estimate displacements in terms of derivatives of an approximating function, as is necessary if one attempts to estimate the displacements by first approximating F. *L. E. Payne.*

Mitra, D. N. Torsion of composite sections of different isotropic materials. Bull. Calcutta Math. Soc. **47** (1955), 191–197.

In this paper a function-theoretic method is adopted to solve the torsion problem of a cross-section which is composed of two or more different isotropic materials, when the cross section can be mapped conformally on the circle and the boundaries of regions of different materials are conformally represented on circumference of concentric circles. Cases when such a cross-section contains a hole are also dealt with. (Author's summary.)

R. M. Morris (Cardiff).

Haringx, J. A. Non-linearity of corrugated diaphragms. Appl. Sci. Res. A. **6** (1956), 45–52.

Proceeding from the method of calculation for determining the rigidity of corrugated diaphragms given in a former paper [Appl. Sci. Res. A3 (1950), 299–325] one is able to indicate what degree of non-linearity of the relations between load and deflection is to be expected for large deformations. By means of an example it is shown that the introduction of the corrugations into the flat plate, though unavoidably increasing the initial rigidity, involves an important gain in maximum deflection for the same degree of non-linearity.

R. Gran Olsson.

Pearson, Carl E. Remarks on the centre of shear. Z. Angew. Math. Mech. **36** (1956), 94–96. (German, French and Russian summaries)

The author examines the two usual methods of defining the center of shear [see, e.g., E. Trefftz, Z. Angew. Math. Mech. **15** (1935), 220–225; W. R. Osgood, J. Appl. Mech. **10** (1943), A-62–A-64; MR **5**, 84]. He gives a simple expression for the discrepancy in the position of the center of shear which arises from the two different methods. This expression involves only the geometry of the region, the elastic constants and the solution to the corresponding torsion problem.

L. E. Payne (College Park, Md.).

Berry, D. S.; and Hunter, S. C. The propagation of dynamic stresses in visco-elastic rods. J. Mech. Phys. Solids **4** (1956), 72–95.

The paper gives a mathematical treatment of stress propagation in visco-elastic rods. Strains are supposed to be infinitesimal; in consequence the material is linear and the relations between stress and strain are given by Boltzmann's principle of superposition. The effect on the longitudinal motion of the inertia of the associated lateral motion is neglected; this restricts validity of the treatment to thin rods. It is assumed that there are no body forces acting on the rod. Under these conditions the partial differential equation of motion is established; a variety of boundary conditions is formulated applying to finite and semi-infinite rods. The general solutions of the equation of motion for transient and for steady state are obtained with a Laplace transform method. Finally explicit expressions are obtained for particular materials characterized by some of the currently employed forms of creep or relaxation function (Maxwell solid, Voigt solid, three parameter body, a form of power law). The inversion of the Laplace integrals necessary for obtaining the explicit solutions is performed by the complex inversion formula and contour integration. *B. Gross* (Rio de Janeiro).

Weidenhammer, F. Das Stabilitätsverhalten der nichtlinearen Biegeschwingungen des axial pulsierend belasteten Stabes. Ing.-Arch. **24** (1956), 53–68.

The author studies transverse vibrations in a rod subject to an axial force oscillating harmonically. In certain frequency ranges such oscillations appear and are limited in magnitude by the nonlinear nature of the mechanical system. The basic mechanical equations are written down, a system of ordinary differential equations in quasi-normal mode form being obtained. It is shown that the coupling between these equations is small. They are then solved as single second order equations of nonlinear Mathieu type. The methods of van der Pol and Kryloff-Bogoliuboff are used. The stability of the oscillations is investigated. Extensive numerical calculations are represented graphically. *E. Pinney* (Berkeley, Calif.).

Dengler, Max A. Transversale Wellen in Stäben und Platten unter stossförmiger Belastung. Österreich. Ing.-Arch. **10** (1956), 39–66.

The first part of the paper is concerned with the Laplace transform solution for impact on a beam, with transverse shear deformation and rotatory inertia taken into account. Some numerical results are given for the time variation of bending moments at three selected stations along the span. The second part of the paper outlines an approach to the corresponding problem for rotationally symmetric impact on an infinite plate, without however showing the practicality of this approach.

E. Reissner (Cambridge, Mass.).

Morrison, J. A. Wave propagation in rods of Voigt material and visco-elastic materials with three-parameter models. Quart. Appl. Math. **14** (1956), 153–169.

Using simple, linear, one-dimensional theories of visco-elasticity, the author obtains solutions for a semi-infinite rod with constant velocity or stress applied impulsively and maintained at the end. Graphs of stress and strain distributions for these problems are given.

J. L. Ericksen (Washington, D.C.).

Weidenhammer, F. Stabquerschwingungen schwach vorgekrümmter Stäbe mit pulsierender Axiallast. Z. Angew. Math. Mech. **36** (1956), 235–238.

Vernotte, Pierre. Le problème de la barre en régime périodique quelconque, les propriétés physiques dépendant de la température. C. R. Acad. Sci. Paris **242** (1956), 2913–2915.

Sengupta, A. M. Stress distributions in a thin plate around a hole in the form of a loop of lemniscate of Bernoulli. Bull. Calcutta Math. Soc. **47** (1955), 153–156.

Gumenyuk, V. S. On free vibrations of plates of varying thickness. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 130–133. (Ukrainian. Russian summary)

The usual method for finding the frequencies of free vibrations (eigenvalues) when a differential equation is approximated by a system of difference equations, is to set the determinant of the system equal to zero and find the roots of the resulting polynomial equation. The author uses a similar method for finding frequencies of the first mode for plates of thickness varying along one of the coordinates. He applies rectangular grids for rectangular plates and tirangular grids for triangular plates. In all

cases the plates are freely supported on the boundaries. Plates of constant thickness become special cases. For a square plate of constant thickness the author's value differs by 4% from the exact value. *T. Leser.*

Tersenov, S. A. On the asymptotic behavior of the eigenvalues and eigenfunctions of vibration of thin sloping shells. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 583–589. (Russian)

The physical problem in question leads to an elliptic system of differential equations for the displacement vector. Asymptotic formulas for the eigenvalues and eigenfunctions of the corresponding vibration problem are given with the aid of the methods of Carleman [Åttonde Skand. Matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 34–44] and Å. Pleijel [Amer. J. Math. **70** (1948), 892–907; MR **10**, 301]. *L. Gårding.*

Chatterjee, B. B. Stresses in certain thin blades rotating about an axis lying in their middle plane. Z. Angew. Math. Mech. **36** (1956), 231–233.

Helfenstein, Heinz G. Graphical determination of a discontinuity surface by wave reflection. Quart. Appl. Math. **14** (1956), 93–97.

It is assumed that the seismic disturbances follow simple ray theory. Measurements are made at the surface of the earth. A geometric construction is given for locating an underlying reflecting surface from a few such measurements. The application of the theory to practical situations, particularly ones involving several reflecting layers, is not discussed. *E. Pinney* (Berkeley, Calif.).

Adachi, Ryuzo. On a plane wave propagated in an elastic plate with infinite breadth. Kumamoto J. Sci. Ser. A. **2** (1955), 184–195.

A wave is assumed to propagate in a single direction parallel to the face of an elastic plate and within that plate. The elastic wave equations and the boundary conditions yield transcendental equations from which the velocity of propagation of the assumed wave may be determined. In general a finite number of propagating modes exist depending upon the frequency, plate thickness, and P- and S-wave velocities; the various cases are indicated. *E. Pinney* (Berkeley, Calif.).

Satô, Yasuo. Analysis of dispersed surface waves by means of Fourier transform. II. Synthesis of the movement near the origin. Bull. Earthquake Res. Inst. Tokyo **34** (1956), 9–18. (Japanese summary)

Nagase, Masahumi. Diffraction of elastic waves by a spherical surface. J. Phys. Soc. Japan **11** (1956), 279–301.

Elastic waves emanate from a dilational or distortional source of dipole type near a spherical cavity in an infinite elastic medium. The straightforward solution in terms of Bessel functions and Legendre functions is written down. The bulk of the paper is devoted to applying Watson's method to transforming this, in the several cases, to more rapidly converging series, better adapted to numerical calculation. This difficult task is the most important contribution of the paper and has cost the author much labor. *E. Pinney* (Berkeley, Calif.).

Makarov, G. I.; and Petrašen', G. I. Unsteady diffraction of acoustic and electromagnetic waves from a sphere. Leningrad. Gos. Univ. Uč. Zap. **170.** Ser. Mat. Nauk **27** (1953), 266–301. (Russian)

Since a steady-state solution does not yield a satisfactory approximation in case of diffraction of a pulse the latter is considered for a spherical obstacle and a plane incident wave having the character of a unit pulse or of Dirac δ-function. No essential changes are required for the case of a spherical wave. The discussion of a wave equation is made simultaneously for acoustics (solid sphere) and electromagnetism (conducting sphere). Each potential (inside or outside of the sphere) is given in the form of a series of Legendre polynomials with coefficients represented by some Mellin contour integrals. A formula is derived showing the shape and position of wave fronts for incident and reflected waves. *W. Jardetzky.*

Petrašen', G. I.; Smirnova, N. S.; and Gel'činskiĭ, B. Ya. Some problems in the dynamical theory of elasticity for media containing cylindrical or spherical boundaries of separation. Leningrad. Gos. Univ. Uč. Zap. **170.** Ser. Mat. Nauk **27** (1953), 221–265. (Russian)

The problems concerning the cylinders are considered by the second author, those concerning the sphere by the third author both under the guidance of the first author [see the preceding review]. In a two-dimensional problem the radial component of the stress at the cylindrical surface is a function of the azimuth and time. No concentrated forces are considered. The action of a line source inside the cylinder is the second problem treated in this paper. Few remarks are made about the case of a concentrated force at a point of the surface and about a two-layered cylinder. Methods of the first author [see the following review] are applied then in the discussion of point sources in an unlimited medium and in a solid sphere or a two-layered sphere. *W. Jardetzky.*

Petrašen', G. I. Dynamical problems of the theory of elasticity in the case of an isotropic sphere. I. Free and forced vibrations of an elastic sphere. Leningrad. Gos. Univ. Uč. Zap. **135.** Ser. Mat. Nauk **21** (1950), 24–70. (Russian)

In a previous paper [same Zap. **17** (1949)] the author considered a system of "spherical vectors" analogous to a system of spherical functions. A solution of Lamé equations for a sphere can be taken in the form of a series of such spherical vectors. It is assumed that the initial value of every function (displacement or stress) is equal to zero and that the boundary value is expanded in a series of spherical vectors. The boundary conditions are considered for the case of the functions of the radius and time. Two methods are applied to solve the problems, namely, the method of Smirnoff using the sources for representation of a function and Laplace transforms. The second method seems to be more convenient. *W. Jardetzky.*

Knopoff, L. Diffraction of elastic waves. J. Acoust. Soc. Amer. **28** (1956), 217–229.

The Kirchoff theory of the retarded potential is modified to treat the vector partial differential equation satisfied by the displacement vector in an elastic medium. Green's functions for this problem are not discussed. The theory is applied to obtain some general results in the diffraction of P- and S-waves. *E. Pinney* (Berkeley, Calif.).

★ **Sokolovskij, V. V. Theorie der Plastizität.** VEB Verlag Technik, Berlin, 1955. 484 pp. DM 28.40. Translation of: Teoriya plastičnosti. 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. Reviewed in MR **13**, 185.

See also: Mel'nik, p. 1190; Pekeris, p. 1216.

MATHEMATICAL PHYSICS

★ **Vogel, Théodore. Physique mathématique classique.** Librairie Armand Colin, Paris, 1956. 214 pp.

The author distinguishes in his introduction between theoretical physics and mathematical physics. Mathematical physics consists, in his view, in reducing physics to an axiomatic form; it bears roughly the same relation to theoretical physics as Euclid's Elements of Geometry did to the rules of mensuration of his predecessors.

Although the aim of the book is the axiomatisation of physics, only the least chapter (pp. 147–208), is devoted to this; the rest is concerned with explaining the mathematical tools used and the properties of the integral and partial differential equations which occur. This part is largely a statement of results with little or no proof.

The conclusion is: — „La physique mathématique classique est l'étude des solutions du problème d'extremum:

$$\delta \int [cT(\varphi)\cdot T(\varphi) - U\cdot\varphi]d\tau = 0,$$

où c, U, φ, T sont des fonctions tensorielles dans l'espace-temps, c et U sont données, φ à déterminer par la condition d'extremum, T donné en fonction de φ."

E. T. Copson (St. Andrews).

★ **Bauer, Edmond. Champs de vecteurs et de tenseurs. Introduction à l'électro-magnétisme.** Masson et Cie, Paris, 1955. vii+204 pp. 2200 francs.

Professor Bauer's book gave this reviewer a quite unexpected urge to teach again a course in vector and tensor analysis. The author's basic idea is that the understanding of physics, the clear distinction in the student's mind between experimental law, theoretical hypothesis, and mathematical identity calls for a precise formulation of the mathematical language of physics. In treating vectors and tensors from this point of view, he succeeds not only in making his subject admirably clear, but also in making it truly interesting. The first three chapters devoted to vectors, tensors, and their fields appeal repeatedly to the reader's geometrical and physical intuition. Chapter IV on tensors in an arbitrary space, generalizes and formalizes the basic concepts and methods and leads to the idea of abstract space. The last chapter on applications to the electromagnetic field presents an outline of a course on the subject. It is somewhat debatable whether such a skeleton treatment is of much practical use. In view of the general excellence of this graduate level text, it is of interest to note that the first four chapters are based on a course which the author taught at the Ecole Normale Supérieure de Sèvres, a French teachers college for women, graduation from which is required from a woman teaching in the French secondary school system.

J. E. Rosenthal (Passaic, N.J.).

Renaud, Paul; Joly, Maurice; et Dervichian, Dikran G. Sur certaines hypothèses simplificatrices qui engendrent la notion de conservation de l'énergie dans les domaines où elles sont applicables. C. R. Acad. Sci. Paris **243** (1956), 24–26.

See also: Davies, p. 1214; Beach and Theus, p. 1221; Berger, p. 1221.

Optics, Electromagnetic Theory, Circuits

Kubo, Kazuko. On the intensity of light scattered by fine particles. J. Sci. Res. Inst. Tokyo **50** (1956), 10-18.

The behavior of light-scattering, varying with particle concentration, is studied on the basis of multiple scattering. The angular intensity distribution is represented by Legendre's polynomial series and successively determined the fluxes of scattered light. *Author's summary.*

Lenoble, Jacqueline. Application de la méthode de Chandrasekhar à l'étude du rayonnement diffusé dans le brouillard et dans la mer. Rev. Opt. **35** (1956), 1–17.

The general solution for the problem of diffuse reflection and transmission by a plane parallel atmosphere scattering light in accordance with a phase function, $p(\cos\Theta)$, expressible as a series in Legendre polynomials has been given by Chandrasekhar [Radiative transfer, Oxford, 1950, pp. 149–159; MR **13**, 136] in a method of approximation in which the appropriate equation of transfer is systematically replaced by systems of equations of increasing order. The relevant formulae of this solution are summarized and are applied to the illumination of the sea. For this case the phase function

$$p(\cos\Theta) = \varpi_0[1 + 1\cdot 73P_1(\cos\Theta) + P_2(\cos\Theta)],$$

where ϖ_0 is the albedo for single scattering. The explicit forms of the solution are obtained for the cases $\varpi_0 = 1.00$, 0.66, 0.50 and 0.33 and are applied to an ocean of infinite depth when illuminated by a uniform sky and when the sun is at a zenith distance of 59°. The case when a layer of mist (of albedo 1), is superposed on the sea and the illumination is by a uniform sky is also considered. The results are illustrated by numerous graphs.

S. Chandrasekhar (Williams Bay, Wis.).

Gross, Eugene P.; and Ziering, Sigi. Theory of transfer processes. Astrophys. J. **123** (1956), 343–352.

The solution of the equation of radiative transfer for plane parallel atmospheres by expanding the intensity $I(\tau, \mu)$ as a series in Legendre polynomials is well-known. In order that the conditions at the bounding surfaces (namely, that the intensity vanishes on a hemisphere) may be more satisfactorily met than in the usual methods, the author expands the intensity in the outward $(1 \geqq \mu \geqq 0)$ and the inward $(-1 \leqq \mu \leqq 0)$ directions as separate series, thus:

$$I(\tau, \mu) = \sum_n B_n^+(\tau) P_n^+(\mu) \qquad (1 \geqq \mu \geqq 0),$$

$$I(\tau, \mu) = \sum_n B_n^-(\tau) P_n^-(\mu) \qquad (-1 \leqq \mu \leqq 0),$$

where

$$P_n^+(\mu) = P_n(\mu)(2\mu - 1) \text{ and } P_n^-(\mu) = P_n(\mu)(2\mu + 1).$$

These functions $P_n^{\pm}$ form a complete set of orthogonal functions in the ranges (0, 1) and (−1, 0), respectively. (These functions were apparently first introduced by J. Yvon [cf. J. B. Sykes, Monthly Not. Roy. Astr. Soc. **111** (1951), 377–386; MR **13**, 874].) With the intensity ex-

pressed in this fashion, the equation of transfer leads to a system of coupled linear equations with constant coefficients for the $B_n^{\pm}$'s; these equations can be solved by elementary methods in any finite approximation. The solutions for the case of isotropic scattering and scattering in accordance with Rayleigh's phase function are obtained by this method in the first two approximations. The results thus obtained are compared with those obtained by the other methods and with the exact results when these are available. [A method essentially equivalent to the authors' has been used earlier by Sykes, loc. cit.]

S. Chandrasekhar (Williams Bay, Wis.).

★**Damien, René. Théorème sur les surfaces d'onde en optique géométrique. Avec une note sur le miroir intégral.** Gauthier-Villars, Paris, 1955. iv+34 pp. (21 plates). 900 francs.

The author develops an interesting theorem. The rays from a luminous point form, after reflection or refraction at a surface g, a normal system according to the law of Malus. The specific wave surface h is determined by the fact that the light path from S to h is zero. The theorem now states that the surface h' inverse to h with respect to S can be considered as refracting the rays from S in such a way that the surface g' inverse to g with respect to S is a wave surface of the refracted bundle such that the light path between S and g' equals zero.

The theorem is applied to a lot of special problems in order to construct wave surfaces for reflection and refraction at a single special surface.

In the appendix the author suggests a mirror consisting of a set of parabolic mirrors with the same focal point but different focal lengths to be used as a searchlight, which has not a blinding effect. *M. Herzberger.*

Mohorovičić, S. Beiträge zur optischen Abbildung. II. Optik **13** (1956), 224–226.

Geometrical derivations are given for Snell's law of refraction, and also for Hackl's "fictitious" refraction law [Optik **11** (1954), 468–477]. The latter is an approximate form of the refraction law involving the replacement of the surface by its vertex plane. A relation is derived which may be helpful in making the Hackl construction of the refracted ray. *E. W. Marchand.*

Mertz, Lawrence. Optical Fourier synthesizer. J. Opt. Soc. Amer. **46** (1956), 548–551.

Vander Sluis, K. L. General formulas for least squares reduction of Zeeman data. J. Opt. Soc. Amer. **46** (1956), 605–608.

Picht, Johannes. Zur Frage der optischen Lichtweg-Längen zweier Strahlen zwischen Objektpunkt und bildseitigem Schnittpunkt beider Strahlen. Opt. Acta **3** (1956), 1–9.

Expressions are obtained for the optical path difference in two cases: (i) an axial image point in the presence of spherical aberration, (ii) a non-axial image point in the presence of coma. *G. L. Walker.*

Kogan, S. H. On the theory of helical lines. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 541–544. (Russian)

The electromagnetic field distribution in an infinitely long and thin cylinder of infinite conductivity, which is cut up into a spiral, is computed by separation into cylindrical-spiral coordinates. Two extreme cases are considered in particular, i.e., 1. relative width of cut-out $\ll 1$, 2. relative width of conducting band $\ll 1$. Boundary and continuity conditions for the field components are satisfied on the basis of, 1. electrostatic field distribution in the gap, 2. quasistationary current density distribution on the band, both justifiable for low-order modes, if spiral pitch $\ll$ circumference. Approximate expressions for the dispersion characteristics of the lowest modes are obtained. They are plotted for the range $kr \lesssim 1.5$. It turns out that the validity ranges of the expressions for the two limiting cases overlap and cover the whole range of relative widths with acceptable accuracy.

H. G. Baerwald (Cleveland, Ohio).

Durand, Émile. Les équations fondamentales d'un électromagnétisme classique non conservatif. C. R. Acad. Sci. Paris **242** (1956), 1862–1865.

A more elementary treatment is given of an earlier paper [same C. R. **225** (1947), 567–569; MR **9**, 166] which contained a generalization of Maxwell's equations inspired by Dirac's matrix representations. These equations correspond to a non-conservative electromagnetic system where electric and magnetic charges can not only move but also be created or annihilated. In the present treatment a concrete interpretation is given of all the quantities involved. *J. E. Rosenthal* (Passaic, N.J.).

Debever, R.; et Géhéniau, J. Sur la formulation de Langevin des notions et des lois du champ électromagnétique. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 346–355.

Following ideas developed by Paul Langevin in lectures at the Collège de France, the authors define an electromagnetic field by two skew-symmetric tensors, F_{rs} and K^{rs}, in a 4-space which, in general, has no other structure. Then the two alternating forms

$$\delta A = qF_{rs}dx^r\delta x^s, \quad \delta Q = \tfrac{1}{2}K^{tu}\varepsilon_{turs}dx^r\delta x^s,$$

where q is an element of charge, are connected respectively with (i) the force exerted on an elementary charge q having dx^r for an element of its world line, and (ii) electricity induced on a surface element defined by dx^r, δx^s, and magnetic force. *J. L. Synge* (Dublin).

Wait, James R. Radiation from an electric dipole in the presence of a corrugated cylinder. Appl. Sci. Res. B. **6** (1956), 117–123.

The electromagnetic field, **E**, **H**, of a dipole situated near an infinite circular cylinder, whose axis is parallel to that of the dipole, is calculated under the assumption that at the surface of the cylinder the field components satisfy the conditions $E_z = \eta_1 H_\phi$, $E_\phi = -\eta_2 H_z$, where η_1 and η_2 are longitudinal and transverse impedance, respectively. Application is made to the case of a cylinder with periodic circumferential grooves, for which agreement of the solution with experiment is claimed. *R. N. Goss.*

Epstein, Paul S. Theory of wave propagation in a gyromagnetic medium. Rev. Mod. Phys. **28** (1956), 3–17.

A review on the physics and mathematics of those fascinating ferromagnetic materials called ferrites.

C. J. Bouwkamp (Eindhoven).

Pachner, Jaroslav. A theory of the diffraction of an electromagnetic wave by a plane screen. Czechoslovak J. Phys. **6** (1956), 8–12. (Russian summary)

Kear, George. Forward scattering of high-frequency plane waves by a sphere. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-86** (1955), i+29 pp.

Let u denote the total field, u_i the incident plane wave, and u_s the scattered field. The author starts with a representation of $u_s=u-u_i$ where u is an infinite series of radial eigenfunctions, and u_i a contour integral. By appropriately deforming the contour integral, by applying the Hankel approximation to the coefficients of the first terms of the infinite series, the tangent approximation to the coefficients of the remaining terms, and standard asymptotic expression for cylinder functions, the author obtains an asymptotic expression for the scattered wave valid if ka is large (k is wave number, a is radius of sphere). By evaluating this expression in the forward direction, he obtains the scattering cross-section: $\sigma=2\pi a^2\{1+C(ka)^{-3/2}\}$ with $C=0.99668$ for the Dirichlet case and $C=-0.8654$ for the Neumann case. *C. J. Bouwkamp* (Eindhoven).

Levine, Harold. Diffraction by a circular aperture at high frequencies. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-84** (1955), ii+60 pp.

The author investigates the first correction to the plane wave geometrical transmission cross-section of a circular aperture in an infinite plane screen (normal incidence, acoustically soft screen). He predicts a cross-section

$$\sigma\simeq\pi a^2\Big[1-\frac{2}{\pi^{1/2}}\frac{\sin(2ka-\frac{1}{4}\pi)}{(ka)^{3/2}}\Big]\ (ka\to\infty),$$

where a is the aperture radius. It is shown that the straight-edge screen distribution due to a grating plane wave plays the decisive role for the correction in the cross-section. Emphasis is placed on the role of the boundary conditions at the aperture rim. (From the author's abstract.) *C. J. Bouwkamp* (Eindhoven).

Costa de Beauregard, Olivier. Diffraction par une ouverture plane à contour variable. Formules générales. C. R. Acad. Sci. Paris **242** (1956), 347–350.

Vaĭnshteĭn, L. A. Electron waves in retarding systems. I. General theory. Ž. Tehn. Fiz. **26** (1956), 126–140. (Russian)

A linear theory is given of the propagation of waves — in the presence of an electron beam — in various retarding systems, such as transmission lines with slow electromagnetic waves. An analysis of these "electron waves", which is of special interest in the theory of the travelling wave tube, is carried out under the following assumptions: small signals varying harmonically with time; monochromatic electron beam; retarding systems that are either homogeneous or consist of periodic structures such that the inhomogeneity does not affect the forces acting on the electron beam. The treatment is based on the theory of excitation of wave guides and on a variational method. The characteristic equation of electron waves is derived and its solutions are investigated. The physical meaning of quantities entering in the characteristic equation is considered. *J. E. Rosenthal.*

Vaĭnshteĭn, L. A. Electron waves in retarding systems. II. Concrete problems. Ž. Tehn. Fiz. **26** (1956), 141–148. (Russian)

The general theory developed in the paper reviewed above is applied to the investigation of electron waves in particular types of retarding systems, such as a wave guide with a dielectric, with a fin structure, and a spiral wave guide. Both hollow and solid electron beams are considered. A comparison is made with theoretical results to be found in the literature. *J. E. Rosenthal.*

Nardini, Renato. Osservazioni su una relazione energetica della magneto-idrodinamica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 376–377.

The author points out that the equation one generally writes down as expressing the first law of thermodynamics (namely, $dQ+dL=dT+dU$, where dT and dU are the increase in the kinetic and the internal energies consequent to a supply of heat dQ and an amount of mechanical work dL done by external forces) can be generalized in the framework of hydromagnetics by including in dL the work done by the impressed electric field, the Poynting flux of electromagnetic energy and the ohmic-Joule dissipation, and also including the magnetic energy $(H^2/8\pi)$ as a part of internal energy. The reason for this is, that the volume change in the kinetic energy resulting from the term $\mathbf{j}\times\mathbf{H}$ in the equation for $\mathbf{v}$ is exactly balanced by the corresponding change in the magnetic energy resulting from the term $\mathbf{v}\times\mathbf{H}$ in the equation for $\mathbf{H}$. *S. Chandrasekhar* (Williams Bay, Wis.).

Zeuli, Tino. Oscillazioni magneto-idrodinamiche in una massa liquida elissoidale rotante elettricamente conduttrice. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89** (1954–55), 270–285.

Hydromagnetic oscillations about Agostinelli's stationary solution [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1954), 216–221; MR **17**, 542] is considered. This paper is an extension of similar investigations by Agostinelli [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89** (1955), 41–58, 68–92; MR **17**, 543]; the mathematical reductions are too complicated to summarize. *S. Chandrasekhar* (Williams Bay, Wis.).

★ **Okada, S. Algebraic and topological foundations of network synthesis.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 283–322. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

This is a general and at times quite mystical discourse on the concepts and methods of network theory. *R. Bott* (Princeton, N.J.).

Linsman, M. Sur une méthode de simplification des circuits dans la théorie des fonctions de commutation. Bull. Soc. Roy. Sci. Liège **25** (1956), 163–166.

This paper discusses the application of Boolean algebra to electronic circuits, in particular the "minimizing heart" of H. H. Aiken [Synthesis of electronic computing and control circuits, Harvard, 1951, Ch. V; MR **13**, 497]. *R. W. Hamming* (Murray Hill, N.J.).

Belevitch, V. Les configurations des circuits de redresseurs. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 372–378.

Configurations of circuits containing rectifiers are discussed from a geometrical viewpoint. *C. Y. Lee.*

Gavrilov, M. A. Relay schemes with vacuum-tube circuits. Avtomat. i Telemeh. **16** (1955), 328–343. (Russian)

This paper discusses the reduction of the number of

relay contacts in multi-input-switch, multi-output-coil networks by the use of diodes. It is stated and exemplified that by the insertion of resistance and (usually) many rectifiers, such a network can be synthesized in a manner using exactly one transfer contact on each input variable.
V. E. Beneš (Murray Hill, N.J.).

Ivanov, V. I. Cyclic relay circuits and analytic relations in them. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 239–241. (Russian)

The author states two trivial theorems concerning sequential relay circuits having periodic behaviour such that each relay operates and releases once per period in a fixed sequence. Theorem 1 states in part that, if the relays are assigned indexes in order of their operation, and the sequence of operation is such that whenever the ith relay is operated, all prior relays (those having indexes less than i) are operated, then in any two-terminal contact network which is in series with a front contact on relay i, front contacts on all prior relays may be replaced by short circuits. The other parts of the theorem are obvious modifications of this statements. Theorem 2 deals with circuits in which only one relay at a time is operated, and states a number of direct consequences of this property in terms of Boolean algebra. *V. E. Beneš.*

Bell, D. A. Application of equipartition theory to electric circuits. Proc. I.R.E. **44** (1956), 1065.

Mostafa, Abd El-Samie. Generalised simple solutions for circuits with periodically varying parameters. I. Proc. Math. Phys. Soc. Egypt **5** (1954), no. 2, 27–49 (1955).

A simple resonant circuit is considered, containing a harmonically oscillating capacitance. The charge on the condenser satisfies Mathieu's equation. The treatment of Mathieu's equation by the Floquet and Hill theories is noted and judged computationally impractical.

When the amplitude of the capacitance oscillation is small and the natural frequency of the circuit is near that of the oscillation, the author solves the equation using a perturbation technique. This is elaborated to treat the case when the capacitance oscillation is not small. These techniques are based on heuristic circuit theory arguments. It is not demonstrated that the successive approximations thus arrived at properly converge to the solution of the equation. *E. Pinney.*

Mostafa, Abd El-Samie. Generalised simple solutions for circuits with periodically varying parameters. II. Periodically varying inductance oscillatory circuits. Proc. Math. Phys. Soc. Egypt. **5** (1954), no. 2, 63–73 (1955).

This paper treats a series circuit consisting of a time-dependent inductance $L(1+g\cos 2\omega t)$ and a constant capacitance, the problem being of practical importance in analysis of frequency modulated waves and warble tones. Details of the method are apparently in Part I [see the preceding review] and the author claims that the mathematics is elementary. Numerical results obtained by the method for a specific case are compared with those from a differential analyzer [E. I. Hawthorne, Proc. I.R.E. **39** (1951), 78–81], excellent agreement being obtained. *H. D. Conway* (Ithaca, N.Y.).

See also: Watanabe, p. 1179; Makarov and Petrašen', p. 1255.

Quantum Mechanics

Popovici, Andrei. Wave and particle. Gaz. Mat. Fiz. Ser. A. **8** (1956), 127–138. (Romanian)

A survey article on the physical interpretation of wave mechanics. It has a sixty-eight item bibliography ranging Einstein to Lenin. *M. J. Moravcsik* (Upton, N.Y.).

Hylleraas, Egil A. Zur praktischen Lösung der relativistischen Einelektronengleichung. Z. Physik **140** (1955), 626–631.

The Dirac equation for the hydrogen atom is solved in a very elegant and condensed form by separating the large and small components and expressing the solutions of the resulting second-order equations in terms of Laguerre-functions. The author's derivation gives a considerable simplification of the treatment of this fundamental problem. *P.-O. Löwdin* (Uppsala).

Tietz, T. Note to the solution of the Schrödinger equation for finite systems. Nuovo Cimento (9) **12** (1954), 449–451.

The author determines the number of eigenvalues of the Schrödinger equation for finite systems by considering the zeros of the hypergeometric functions.
P.-O. Löwdin (Uppsala).

Tietz, T. Über die physikalisch brauchbaren Randbedingungen der Schrödinger-Gleichung. Ann. Physik (6) **15** (1954), 79–81.

A fundamental problem in quantum mechanics is to find the proper boundary conditions to impose on the Schrödinger equation in order to select the physically meaningful wave functions. The author points out that, for the simple case of the hydrogen atom, the condition of quadratic integrability is not sufficient and that a requirement that the eigenfunctions should be self-adjoint leads to a more uniform treatment of the physical solutions for allquantum numbers.
P.-O. Löwdin (Uppsala).

Lentei, I. Die komplexen Lösungen der Schrödinger-Gleichung und die WKB-Methode. Acta Phys. Acad. Sci. Hungar. **5** (1955), 353–356.

The divergency difficulty of the WKB-method is investigated. By transforming the exact solutions of the Schrödinger equation for the ground states of the harmonic oscillator and the hydrogen atom to WKB-form, the author has shown that the solutions have an essential singularity for $h=0$ and that a power series expansion in h thus does not exist. In this connection the exact eigenvalue condition is also discussed. *P.-O. Löwdin.*

Sucher, J. Reduction of the Dirac equation. Phys. Rev. (2) **103** (1956), 468.

A new method for reducing to large components is given of an electron in an external electromagnetic field. This method yield equations linear in the energy operator and hence the stationary state problem is a linear eigenproblem. The only other linear reduction given so far is the Foldy-Wouthuysen transformation [Phys. Rev. (2) **78** (1950), 29] which however cannot be exhibited in closed form. The method given in this paper is in closed form. The wave function is written as the sum of two parts, ϕ_1 and ϕ_2, which are eigenfunctions of the projection operators on the positive and negative energy parts, respectively. If then each of the ϕ_i's are in turn written as

the sum of two terms ϕ_i^+ and ϕ_i^- which are $\frac{1}{2}(1\pm\beta)$ times the corresponding ϕ_i, then there is a simple relationship between ϕ_i^+ and ϕ_i^-. The original eigenvalue equation can therefore be written purely in terms of ϕ_i^+ if we make a given alteration in the original Hamiltonian. For a stationary problem ϕ_2^+ is zero, so that the Dirac equation contains only ϕ_1^+. The new Hamiltonian is not completely Hermitian, but for the hydrogen atom problem it is Hermitian to at least α^2. *M. J. Moravcsik.*

Kastler, Daniel. Algèbre multilinéaire et quantification du champ de Dirac. C. R. Acad. Sci. Paris **242** (1956), 2805–2808.

The explicit quantization of the Dirac electron field, along the lines originated by Cook [Trans. Amer. Math. Soc. **74** (1953), 222–245; MR **14**, 825], and parallel to the author's work on the photon [cf. C. R. Acad. Sci. Paris **242** (1956), 2445–2448; MR **17**, 1166]. *I. E. Segal.*

Polievktov-Nikoladze, N. M. Renormalization of charge without perturbation theory. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 458–460. (Russian)

An outline is given of a calculation in which the renormalization of the charge of the electron is carried out without the use of perturbation theory. Use is made of a "generating functional" similar to that obtained in the case of interacting nucleon and pseudoscalar fields [E. S. Fradkin, same Dokl. (N.S.) **98** (1954), 47–50; MR **16**, 317]. The relation between the experimental charge e and the bare charge e_0 which is found is the same as that obtained previously by other authors on the basis of simplifying assumptions or approximate calculations. *N. Rosen.*

Polievktov-Nikoladze, N. M. On the Green's function for a photon. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 703–705. (Russian)

From the "generating functional" of the paper reviewed above a general expression is set up for the Green's function for the photon. An approximate expression is obtained, good to the first order in e^2, where e is the electron charge. The singularity which is present for a certain very high frequency is explained as being due to the approximation used. *N. Rosen* (Haifa).

Bogoljubov, N. N.; and Širkov, D. V. Charge renormalization group in quantum field theory. Nuovo Cimento (10) **3** (1956), 845–863.

The authors start from the very simple observation [cf. Stückelberg and Petermann, Helv. Phys. Acta **26** (1953), 499–520] that no change in the observable consequences of quantum electrodynamics is obtained if the charge e, the two Green's functions G and $D_{\mu\nu}$ and the vertex function Γ_μ are simultaneously transformed according to

$$G\to zG;\ \Gamma_\mu\to z^{-1}\Gamma_\mu,$$
$$D_{\mu\nu}\to z_3D_{\mu\nu};\ e\to z_3^{-\frac{1}{2}}e,$$

where z and z_3 are arbitrary, finite numbers. After such a transformation the Green's functions are no longer normalized to unity at the mass of the physical electron and photon resp. but at some other value λ^2 of the square of the energy momentum vector $\mathfrak{X}$. They must therefore be considered as functions of the three variables $\mathfrak{X}^2/\lambda^2$, m^2/λ^2 and e^2. Using the invariance property mentioned above the authors derive equations for G and $D_{\mu\nu}$. We reproduce the equation for $D_{\mu\nu}$. With the definition

$$D_{\mu\nu}\Big(\frac{\mathfrak{X}^2}{\lambda^2},\frac{m^2}{\lambda^2},e^2\Big)=\frac{1}{i\mathfrak{X}^2}\Big(g_{\mu\nu}-\frac{\mathfrak{X}_\mu\mathfrak{X}_\nu}{\mathfrak{X}^2}\Big)d\Big(\frac{\mathfrak{X}^2}{\lambda^2},\frac{m^2}{\lambda^2},e^2\Big)$$

the equation is

$$\frac{\partial}{\partial x}d(x,y,e^2)=\frac{d(x,y,e^2)}{x}\Big[\frac{\partial}{\partial\xi}d\Big(\xi,\frac{y}{x},e^2d(x,y,e^2)\Big)\Big]_{\xi=1}$$

with the "initial condition" $d(1,y,e^2)=1$. As the authors point out, the "initial condition" at $x=1$ alone is not sufficient to permit an integration of the equation. Instead, one has to know the function $d(x,y,e^2)$ in a finite domain around the point $x=1$ and for all values of y and e^2. The authors then assume that the limit of $d(x,y,e^2)$ exists as $y\to0$ and investigate the resulting theory. Physically, this means that one assumes that the Green's function has a good behaviour in a limit, where an arbitrarily large number of real particles can be created at a finite energy. At this point, no improvement over previous work [Gell-Mann and Low, Phys. Rev. (2), **95** (1954), 1300–1312; MR **16**, 315] is obtained. Finally, the authors show that if perturbation theory is used to estimate the initial values, the resulting asymptotic relations can be integrated and yield an asymptotic form of the Green's functions identical with the results of Landau, Abrikosov, and Halatnikov [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 497–500, 773–776, 1177–1180; **96** (1954), 261–264; MR **16**, 315, 316]. Corresponding problems in the low energy region and pseudoscalar meson theoy are also discussed. *G. Källén* (Copenhagen).

Bogolyubov, N. N.; and Širkov, D. V. Multiplicative renormalization group in quantum field theory. Ž. Eksper. Teoret. Fiz. **30** (1956), 77–86. (Russian)

This paper is an expanded version of the author's earlier work [Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 203–206; MR **17**, 441]. An English translation of the longer version is reviewed above. The points included in this paper which were not in the shorter version are the following. (1) An explicit derivation of the asymptotic relations of M. Gell-Mann and F. Low [Phys. Rev. (2) **95** (1954), 1300–1312; MR **16**, 315] from the author's differential identities. (2) A similar derivation of asymptotic formulas for the Green's functions in the region of low momenta where the "infra- red catastrophe" occurs. *F. J. Dyson* (Princeton, N.J.).

Kato, Tosio. On the eigenfunctions of many-particle systems in quantum mechanics. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **CX-25** (1956), i+42 pp.

In amplification of earlier work of the author [Trans. Amer. Math. Soc. **70** (1951), 195–211; MR **12**, 781] the local properties of eigenfunctions of the Hamiltonian are treated in a rigorous manner. It is shown, using the Green's function of the Laplace operator, that such eigenfunctions are continuous thruout configuration space, and that except at the singularities of the Coulomb-type potential, have partial derivatives, which are bounded. Here 'eigenfunction' means element of the range of the spectral projection corresponding to a finite interval in the spectrum of the Hamiltonian (proved essentially self-adjoint in the cited work), and so includes the conventional eigenfunctions associated with the discrete spectrum, but does not include those functions somewhat similarly associated with the continuous spectrum, except in integrated ('wave packet') form. Near singularities of the potential the eigenfunctions are shown to behave essentially as in the case of the hydrogen atom. *I. E. Segal* (Chicago, Ill.).

Yeivin, Y.; and de Shalit, A. Statistical weights in many-particle systems. Nuovo Cimento (10) **1** (1955), 1146–1151.

A many-particle system consisting of a mixture of particles of spin $\frac{1}{2}$ and spin 1 is considered. By combinatorial methods, the authors have derived a general formula for the statistical weights of all states having a given total spin S. *P.-O. Löwdin* (Uppsala).

Schiff, L. I. Approximation method for high-energy potential scattering. Phys. Rev. (2) **103** (1956), 443–453.

A method is developed to treat high energy potential scattering which is particularly useful in those cases when the partial wave analysis is laborious and the Born Approximation (BA) is not applicable or is not accurate enough. The immediate motivation for this approach was the treatment of high energy electron scattering from heavy non-spherical nuclei. The method uses all terms in the infinite Born series in an approximate way, instead of the first term without an approximation as the BA does. The new method approximates each term in the Born series by using the method of stationary phases, and then sums the series. The applicability of the new method is partly in other domains than the usefulness of the BA. If E', θ, k, and v are the kinetic energy, scattering angle, wave number, and speed of the scattered particle, and V and R are the approximate strength and range of the scattering potential, then the BA holds if $|V|R/\hbar v \ll 1$. The new method imposes no restriction on $|V|R/\hbar v$, but demands that $|V|/E' \ll 1$, and that the potential vary slowly compared to the wavelength. There is also a restriction on the angle: the method can be used either if $\theta \ll (kR)^{\frac{1}{2}}$, or if $\theta \gg (kR)^{\frac{1}{2}}$. The present paper derives formulae for the scattering amplitude in this approximation for the cases of the Schrödinger equation and the Dirac equation, for small and large angles. In order to compare the method with previous treatments, the wavefunction is also discussed at least in the Schrödinger case. It is shown that the accuracy reached by this method is essentially beyond the reach of the WKB method. *M. J. Moravcsik* (Upton, N.Y.).

Gel'fand, I. M.; and Yaglom, A. M. Integration in function spaces and its application to quantum physics. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 77–114. (Russian)

This is a clear expository account of the development of Wiener's mathematical theory of Brownian motion contributed to by Kac, Cameron and Martin, etc., on the one hand, and the work of Feynman and other physicists formulating quantum field theory in terms of integration on function spaces. Emphasis is on the role of the mathematical work in quantum theory, rather than in itself or in relation to stochastic processes. There is a fairly extensive bibliography.

Section 1 develops the theory of Wiener measure in the space of continuous functions, which is related to the heat equation. Attention is given the evaluation of special integrals along the lines represented by the work of Kac [Trans. Amer. Math. Soc. **65** (1949), 1–13; MR **10**, 383]. Section 2 relates functional integration to the problems of quantum mechanics, and especially to the Schrödinger equation. The origin of the approach is apparently the (unpublished) Ph.D. thesis of Feynman, whose well-known papers provide the main physical basis for the present exposition. Section 3 treats transformation of Wiener integrals, including notably work of Cameron and Martin. Section 4 develops applications to quantum theory, and especially to fields. The present theory clarifies and transforms the structure of quantum fields; it does not fundamentally change the theory, which remains divergent. *I. E. Segal* (Chicago, Ill.).

Turner, Robert D. The diffraction of a cylindrical pulse by a half-plane. Quart. Appl. Math. **14** (1956), 63–73.

In the problem of the title, the wave function is constrained to vanish on the half-plane, the excitation being a line source parallel to the edge of the half-plane. The Green's function is obtained by applying the one-sided Laplace transformation and a modified Kontorowich-Lebedev transformation. *C. J. Bouwkamp.*

Wilcox, Calvin H. A generalization of theorems of Rellich and Atkinson. Proc. Amer. Math. Soc. **7** (1956), 271–276.

In extending work of Rellich, Atkinson, and others, the author is concerned with representation, expansion and uniqueness theorems for wave functions describing the scattering of time-harmonic scalar waves by bounded scatterers. *C. J. Bouwkamp* (Eindhoven).

Baijal, J. S.; and Singh, K. K. The energy-levels and transition probabilities for a bounded linear harmonic oscillator. Progr. Theoret. Phys. **14** (1955), 214–224.

The quantum-mechanics for a linear harmonic oscillator within a box is treated, and the first five energy levels and the oscillator strengths for transitions involving the first three states are evaluated by using properties of the confluent hypergeometric functions. It is shown that, if the length l of the box is smaller than the classical amplitude l_0, the particle behaves more or less like a free particle enclosed in the box, whereas, if $l > 3l_0$, the first few energy levels resemble those of a free oscillator. In the intermediate region, the bounded oscillator has properties of particular interest. *P.-O. Löwdin* (Uppsala).

Fukuda, Nobuyuki. Formal theory of nuclear models. Phys. Rev. (2) **103** (1956), 420–425.

As it often happens with physical theories, the theory of the nucleus which started with papers by Brueckner, Watson and others about three years ago has been developing along two different lines. Some workers have taken the theory for granted and tried to develop its consequences, as if seeking an a posteriori justification of the basic idea. Others, however, have been more concerned with formal and abstract aspects of the theory and with its mathematical foundations. The present paper belongs to this second category. It deals with the relationship between the actual nuclear problem and a model problem which is in fact solved instead. The present paper gives the general form of the model operator, i.e. the transformation of the wave function and the Hamiltonian from any model space to the space of the actual physical problem, thus making the problem independent of the actual nature of nuclear forces. In this connection the question of uniqueness of model spaces arises, and a method is wanted to select a model space which yields as many actual properties as possible. The model operator is given by an integral equation, and with appropriate boundary conditions uniqueness is conjectured. It is then shown that the integral equation giving the model operator is equivalent to an equation for the Ω scattering matrix derived by H. Tanaka [Progr. Theoret. Phys. 13

(1955), 497–504] but the boundary conditions are different. Finally the independent particle model is considered as an example and a generalized form of the Hartree-Fock method is given. This generalization contains as a special case the extension of the Hartree-Fock method by Eden and Francis [Phys. Rev. (2) **97** (1955), 1366–1379].

M. J. Moravcsik (Upton, N.Y.).

Plebański, J.; and Sawicki, J. Remarks on the relativistic two-body problem in the classical scalar meson field theory. Acta Phys. Polon. **14** (1955), 455–470. (Russian summary)

This paper describes the relativistic treatment of the motion of two nucleons, interacting through a classical scalar meson field. The authors find that the relativistic result closely agrees with the non-relativistic one, when the distance between the nucleons is sufficiently large. But at small distances the relativistic corrections are found to be very important. The main interest of this investigation seems to be mathematical rather than physical.

S. N. Gupta (Detroit, Mich.).

Morse, Philip M. Waves in a lattice of spherical scatterers. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 276–286.

Kamefuchi, S.; and Umezawa, H. On the limit of applicability of quantum electrodynamics. Nuovo Cimento (10) **3** (1956), 1060–1068.

This is a continuation of an earlier paper [Umezawa, Tomozawa, Konuma, and Kamefuchi, Nuovo Cimento (10) **3** (1956), 772–799; MR **17**, 1165] on the high-energy behavior of renormalizable fields. The authors discuss the relationship between the intrinsic and the observed values of the electronic charge, and it is claimed that the present renormalized quantum electrodynamics breaks down, when it is applied to sufficiently high-energy processes.

S. N. Gupta (Detroit, Mich.).

Kessler, Dan; et Kessler, Paul. Sur les interactions nucléaires du méson mu relativiste. C. R. Acad. Sci. Paris **242** (1956), 3045–3047.

The electromagnetic interaction between μ mesons and nuclei has been calculated on the basis of the semi-classical Weizsäcker-Williams method. This technique results in a formula of the form $\sigma_{m\mu}(E)=\int d\varphi N(\phi, E)\sigma_{ph}(\phi)$ where $\sigma_{m\mu}(E)$ is the cross section for mesons of energy E and $\sigma_{ph}(\phi)$ the corresponding photon cross section. In order to investigate the validity of the semi-classical calculation the authors write down the first order perturbation expression for $\sigma_{m\mu}$ (exchange of a single photon between meson and nucleus) according to quantum electrodynamics. The integral over the final meson momenta is performed approximately; the factor involving the nuclear matrix element then becomes proportional to σ_{ph}. The final result is very close to Weizsäcker-Williams formula, justifying the semi-classical analysis.

R. Arnowitt (Syracuse, N.Y.).

Thermodynamics, Statistical Mechanics

Fujita, Shigeichi. Quasi-stationary process. II. Connected systems and living organism. Kumamoto J. Sci. Ser. A. **2** (1955), 174–183.

[For part I see same J. **2** (1954), no. 1, 51–63; MR **17**, 567.] The linear differential equations that describe the time dependence of fluxes and forces in the neighborhood of equilibrium are solved for the special case of two connected systems. [The case is more special than the author states, because it is not possible, in general, to find such linear combinations of fluxes and forces that all of the conditions he imposes are fulfilled.] The application to "growth and senility of living organisms" seems rather far-fetched.

N. G. van Kampen (Utrecht).

Jeffreys, Harold. The thermodynamics of thermal instability in liquids. Quart. J. Mech. Appl. Math. **9** (1956), 1–5.

The criterion for the onset of instability by convection in a fluid sphere heated within can be derived from a variational principle [Jeffreys and Bland, Monthly Not. Roy. Astr. Soc. Geophys. Suppl. **6** (1951), 148–158; MR **13**, 167; Chandrasekhar, Phil. Mag. (7) **43** (1952), 1317–1329; MR **16**, 639]. The author shows that this principle is equivalent to making extremal the ratio of the energy supplied by expansion to that dissipated by viscosity.

S. Chandrasekhar (Williams Bay, Wis.).

Vertgeĭm, B. A. On conditions giving rise to convection in a binary mixture. Prikl. Mat. Meh. **19** (1955), 745–750. (Russian)

L'auteur étudie quelques propriétés de la convection dans un mélange binaire; on déduit certaines conditions d'équilibre pour un mélange binaire qui est réchauffé d'une façon non uniforme. L'auteur forme les équations des petites perturbations.

En terminant l'auteur étudie le problème de la convection dans un cylindre vertical.

M. Kiveliovitch.

Žuhovickiĭ, E. M. On stability of a nonuniformly heated fluid in a vertical elliptic cylinder. Prikl. Mat. Meh. **19** (1955), 751–755. (Russian)

L'auteur étudie le problème de la convection thermique dans un cylindre vertical elliptique chauffé d'en bas. Pour résoudre le problème l'auteur utilise la méthode de Galerkine.

On trouve que lorsqu'on augmente graduellement le gradient vertical de la température du liquide il se forme un courant antisymétrique: dans une moitié du cylindre le liquide s'élève, dans l'autre il s'abaisse, la séparation est formée par un plan vertical passant par le petit axe de l'ellipse, ce qui est en bonne concordance avec les résultats expérimentaux [V. V. Slavnov, Thèse, Molotov Univ., 1952].

M. Kiveliovitch (Paris).

Klein, Martin J. Generalization of the Ehrenfest urn model. Phys. Rev. (2) **103** (1956), 17–20.

The generalization referred to in the title consists of assuming that if a ball is chosen from urn A there is a probability p that it will be transferred to urn B and if a ball is chosen from urn B there is a probability $p'\neq p$ that it will be transferred to urn A. In the classic problem one has $p=p'=1$. This is suggested as a model of a physical system having two energy levels, represented by the urns. If the system is in contact with a thermal bath at temperature T, then p'/p is assumed to be equal to the Boltzmann factor $\exp(-\varepsilon/kT)$. The mathematical analysis follows closely that of M. Kac [Amer. Math. Monthly **54** (1947), 369–391; MR **9**, 46].

G. Newell.

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